1.2: Twisted Group Algebras and Group Cohomology

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1 Twisted Group Algebras & Group Cohomology

Further in this chapter, we will look at group extensions of a group G by an abelian group Z, which is endowed with a G-action. The parametrisation of these group extensions is given by the second cohomology group $H^2(G; Z)$ - we will define the cohomology groups $H^1(G; Z)$ and $H^2(G; Z)$ in a different light, then show canonical equivalence of these definitions.

Given a group G, field k, and some map $\alpha : G \times G \to k^\times$, we can define the group algebra kG with a 'twisted' k-bilinear multiplication by defining

$$x \cdot y = \alpha(x, y)xy.$$

As a result, the basis G of kG is no longer closed under multiplication. One important note is that this new multiplication isn't necessarily associative. We first give a criteria for associativity.

Theorem 1.1. $\alpha : G \times G \to k^\times$ defines an associative twisted multiplication on kG if and only if

$$\alpha(xy, z)\alpha(x, y) = \alpha(x, yz)\alpha(y, z)$$

for all $x, y, z \in G$.

Proof. Following from definitions and k-bilinearity of multiplication, we have:

$$(x \cdot y) \cdot z = \alpha(x, y)(xy) \cdot z = \alpha(xy, z)\alpha(x, y)xyz$$

On the other hand, we have:

$$x \cdot (y \cdot z) = \alpha(y, z)x \cdot (yz) = \alpha(x, yz)\alpha(y, z)xyz.$$

These are equal if and only if:

$$\alpha(xy, z)\alpha(x, y) = \alpha(x, yz)\alpha(y, z)$$

□

We next introduce 1- and 2-cocycles, which should hint at constructions of more general homology groups. These are introduced with a bit more generality than is necessary for strictly twisted group algebras, but will make stating specific theorems pertaining to twists more concise.

Definition 1.2. Let G be a group, and let Z be an abelian group, written multiplicatively. Let G act on Z from the left, written ${}^x\alpha$, where $x \in G$ and $a \in Z$.

- A **1-cocycle of G with coefficients in Z** is a map $\gamma : G \to Z$ satisfying $\gamma(xy) = \gamma(x)({}^x\gamma(y))$ for all $x, y \in G$. Denote by $Z^1(G; Z)$ the set of all 1-cocycles of G with coefficients in Z.

- A **2-cocycle of G with coefficients in Z** is a map $\alpha : G \times G \to Z$ satisfying the **2-cocycle identity**,

  $$\alpha(xy, z)\alpha(x, y) = \alpha(x, yz)({}^x\alpha(y, z))$$

  for all $x, y, z \in G$. We denote by $Z^2(G; Z)$ the set of all 2-cocycles of G with coefficients in Z.

Note that if G acts trivially on Z, then the 2-cocycle identity is exactly the identity satisfied by α in 1.1. One may check that Z^1 and Z^2 are indeed abelian groups given by pointwise multiplication.

Proposition 1.3. Let G and Z be as in 1.2. Let $\gamma \in Z^1(G; Z)$ and $\alpha \in Z^2(G; Z)$. We have $\gamma(1) = 1$ and for any $x \in G$ we have $\alpha(1, x) = \alpha(1, 1)$ and $\alpha(x, 1) = {}^x\alpha(1, 1)$.

Proof. This follows immediately from the 1- and 2-cocycle properties. □

We now return to the specific example, where $Z = k^\times$.

Definition 1.4. Let G be a group and $\alpha \in Z^2(G; k^\times)$ with $k^\times$ acting trivially on G. The **twisted group algebra of G by α** is the k-algebra, denote $k_\alpha G$, which is equal to kG as a k-module, endowed with the unique k-bilinear product $kG \times kG \to kG$ given by $x \cdot y = \alpha(x, y)xy$ for all $x, y \in G$.

Here, $x \cdot y$ denotes multiplication on $k_\alpha G$, whereas xy denotes the product of x and y in G. Note that in $k_\alpha G$, there is a unit element, but it need not be the unit element 1 of G, as $1 \cdot x = \alpha(1, x)x$. In fact, $\alpha(1, x)$ is independent of x.

Proposition 1.5. Let G be a group. Consider $k^\times$ with the trivial action of G, and let $\alpha \in Z^2(G; k^\times)$. Let $x \in G$.

(i) $\alpha(1, x) = \alpha(1, 1) = \alpha(x, 1)$.

(ii) $\alpha(x, x^{-1}) = \alpha(x^{-1}, x)$.

(iii) The unit element of $k_\alpha G$ is $\alpha(1,1)^{-1}1_G$.

(iv) The multiplicative inverse of x in $k_\alpha G$ is $\alpha(1,1)^{-1}\alpha(x,x^{-1})^{-1}x^{-1}$.

Proof. Statement (i) follows immediately from 1.3. Statement (ii) follows from the 2-cocycle identity, with choices of x, x^{-1}, x we have:

$$\alpha(1,x)\alpha(x,x^{-1}) = \alpha(x,1)\alpha(x^{-1},x).$$

Applying (i) proves (ii). Statements (iii) and (iv) are straightforward verifications, applying multiplication in $k_\alpha G$ and statement (i). □

Proposition 1.6. Let G be a group and $\alpha, \beta \in Z^2(G;k^\times)$. There is a k-algebra isomorphism $k_\alpha G \cong k_\beta G$ mapping $x \in G$ to $\gamma(x)x$ for some scalar $\gamma(x) \in k^\times$ if and only if

$$\alpha(x,y) = \beta(x,y)\gamma(x)\gamma(y)\gamma(xy)^{-1}$$

for all $x, y \in G$.

Proof. Let $x, y \in G$. Write $x \cdot y$ for the product of x and y in $k_\alpha G$ and $x * y$ for the product of x and y in $k_\beta G$. Let ϕ be the isomorphism described - we wish to verify that $\phi(x \cdot y) = \phi(x) * \phi(y)$ for all $x, y \in G$. On one side, we have:

$$\phi(x \cdot y) = \phi(\alpha(x,y)xy) = \alpha(x,y)\gamma(xy)xy.$$

On the other side, we have:

$$\phi(x) * \phi(y) = \gamma(x)\gamma(y)x * y = \gamma(x)\gamma(y)\beta(x,y)xy.$$

These terms coincide if and only if $\alpha(x,y) = \beta(x,y)\gamma(x)\gamma(y)\gamma(xy)^{-1}$, as desired. □

It turns out that 1.6 can be restated in the language of (co)homology. We again return to the more general case, this time to introduce 1- and 2-coboundaries, which form a subgroup of the sets of 1- and 2-cocycles. This will enable us to define the corresponding cohomology groups. Bear in mind the case where $Z = k^\times$ with trivial action for the next part.

Definition 1.7. Let G and Z be as previously stated.

- Denote by $B^1(G;Z)$ the set of maps $\gamma : G \to Z$ for which there exists an element $z \in Z$ for which $\gamma(x) = ({}^x z)z^{-1}$ for all $x \in G$. The elements of $B^1(G;Z)$ are called **1-coboundaries of G with coefficients in Z**. One may show $B^1(G;Z) \trianglelefteq Z^1(G;Z)$. Define $H^1(G;Z) := Z^1(G;Z)/B^1(G;Z)$, the **first cohomology group of G with coefficients in Z**; the elements in $H^1(G;Z)$ are the **first cohomology classes of G with coefficients in Z**.

- Denote by $B^2(G;Z)$ the set of maps $\alpha : G \times G \to Z$ for which there exists a map $\gamma : G \to Z$ such that $\alpha(x,y) = \gamma(x)({}^x\gamma(y))\gamma(xy)^{-1}$ for all $x, y \in G$. The elements of $B^2(G;Z)$ are called **2-coboundaries of G with coefficients in Z**. One may show $B^2(G;Z) \trianglelefteq Z^2(G;Z)$. Define $H^2(G;Z) := Z^2(G;Z)/B^2(G;Z)$, the **second cohomology group of G with coefficients in Z**; the elements in $H^1(G;Z)$ are the **second cohomology classes of G with coefficients in Z**.

We now can restate 1.6 as follows: "There is a k-algebra isomorphism $k_\alpha G \cong k_\beta G$ mapping $x \in G$ to $\gamma(x)x$ for some scalar $\gamma(x) \in k^\times$ if and only if α and β belong to the same cohomology class of G (with coefficients in $k^\times$.)"

We'll perform some basic calculations in the twisted algebra case (where $Z = k^\times$).

Example 1.8.
- If G acts trivially on Z, then it is clear that $B^1(G;Z)$ is trivial. Additionally, $Z^1(G;Z)$ is given by all maps satisfying $\gamma(xy) = \gamma(x)\gamma(y)$, or in other words, all group homomorphisms $G \to Z$. Thus, $H^1(G;Z) = \mathrm{Hom}(G,Z)$. When $Z = k^\times$, $H^1(G;k^\times) = \mathrm{Hom}(G,k^\times)$ is a finite group of order dividing $|G|$. Moreover, if k has characteristic p, and P is a finite p-group, $H^1(G;k^\times)$ is trivial, as $k^\times$ has only one element with order a power of p, 1.
- 1.6 implies that if α represents the trivial class in $H^2(G;k^\times)$, there is a k-algebra isomorphism $kG \cong k_\alpha G$ sending $x \to \gamma(x)x$, and this homomorphism is unique up to a group homomorphism $G \to k^\times$. Hence, *all* such isomorphisms are parameterized by $H^1(G;Z) = \mathrm{Hom}(G,k^\times)$.

Proposition 1.9. Let G be a finite group and $\alpha \in Z^2(G;k^\times)$. The class of α is trivial if and only if $k_\alpha G$ has a module that is isomorphic to k as a k-module.

Proof. First, if $[\alpha] = 1$, then $k_\alpha G \cong kG$, and the trivial kG-module is isomorphic to k as a k-module. Now, suppose $k_\alpha G$ has a module isomorphic to k, call it M. The structural homomorphism of M is the k-algebra homomorphism $k_\alpha G \to \mathrm{End}_k(M)$ given by $x \mapsto (m \mapsto xm)$. However, since M is isomorphic to k as a k-module, $\mathrm{End}_k(M) \cong k$ as k-algebras. (The previous lines came from Robert, he may be able to expand on this) Thus, we have an algebra homomorphism $\varphi : k_\alpha G \to k$. We therefore have that:

$$\alpha(x,y)\varphi(xy) = \varphi(x \cdot y) = \varphi(x)\varphi(y).$$

Hence $\alpha(x,y) = \varphi(x)\varphi(y)\varphi(xy)^{-1}$, and α is a 2-coboundary. □

We may note that $H^i(G;Z)$ for $i = 1,2$ define covariant functors $H^i(G;-) : \mathbf{Ab} \to \mathbf{Ab}$ and contravariant functors $H^i(-;Z) : \mathbf{Grp} \to \mathbf{Ab}$. These functors send $\gamma : Z \to Z'$ to the map $\gamma_* : H^i(G;Z) \to H^i(G;Z')$ defined by post-composition, and send $\varphi : G \to G'$ to the map $\varphi^* : H^i(G';Z) \to H^i*(G;Z)$ defined by pre-composition. In particular, if H is a subgroup of G, then the inclusion of H into G induces restriction maps $\mathrm{res}^G_H : H^i(G;Z) \to H^i(H;Z)$.

Proposition 1.10. Let G be a finite group.

(i) Let Z be a multiplicatively written abelian group on which G acts. Then every element in $H^i(G; Z)$ for $i = 1, 2$ has finite order dividing $|G|$. If Z is finite, then $H^i(G; Z)$ divides both $|G|$ and $|Z|$. In particular, if $\gcd(|G|, |Z|) = 1$, $H^i(G; Z)$ is trivial.

(ii) Let k be an algebraically closed field and consider $k^\times$ with the trivial action of G. Let Z be the group of $|G|$-th roots of unity in $k^\times$. The inclusion $\iota : Z \to k^\times$ induces a surjective group homomorphism $\iota_* : H^2(G; Z) \to H^2(G; k^\times)$. In particular, $H^2(G; k^\times)$ is finite.

(iii) Suppose that k is a perfect field of characteristic p. Let P be a finite p-group. Then $H^2(P; k^\times)$ is trivial.

Proof. (i) Let $\gamma \in Z^1(G; Z)$ and $x, y \in G$. We have $\gamma(xy) = \gamma(x)({}^x\gamma(y))$. Fix x and take the product over all $y \in G$ yields:

$$\prod_{y \in G} \gamma(xy) = \prod_{y \in G} \gamma(y) = \gamma(x)^{|G|} \prod_{y \in G} {}^x\gamma(y).$$

Hence, $\gamma(x)^{|G|} = \prod_{y \in G} \gamma(y)({}^x(\gamma(y)^{-1}))$, implying $\gamma^{|G|}$ is a 1-coboundary and thus every element in $H^1(G; Z)$ has finite order dividing $|G|$. If Z is finite, then so is the set of maps from G to Z, so $H^1(G; Z)$ is finite. Viewing Z as a $\mathbb{Z}$-module, one may see that $|Z|$ annihilates $H^1(G; Z)$ (as it annihilates Z), thus every element in $H^1(G; Z)$ has order dividing $|Z|$ as well.

Now let $\alpha \in Z^2(G; Z)$, and for $x \in G$, set $\mu(x) = \prod_{y \in G} \alpha(x, y)$. Let $x, y, z \in G$ and consider the 2-cocycle identity,

$$\alpha(x, yz)({}^x\alpha(y, z)) = \alpha(xy, z)\alpha(x, y).$$

Taking the product over all z yields:

$$\mu(x)({}^x\mu(y)) = \mu(xy)\alpha(x, y)^{|G|}.$$

This implies that $\alpha^{|G|} \in B^2(G; Z)$, so $\alpha \in H^2(G; Z)$ must have order dividing $|G|$. Similar arguments as before are used when Z is finite to show every element in $H^2(G; Z)$ has order dividing $|Z|$.

(ii) Now let Z be the group of $|G|$-th roots of unity in $k^\times$, and let G act trivially on algebraically closed k. Let $\alpha \in Z^2(G; k^\times)$ and μ as in (i). We wish to find a β in the same cocycle class as α which has values entirely in Z. Since k is algebraically closed, for all $x \in G$, there is $\nu(x) \in k^\times$ for which $\nu(x)^{|G|} = \mu(x)$. Define $\beta(x, y) = \alpha(x, y)\nu(x)^{-1}\nu(y)^{-1}\nu(xy)$ - then β and α belong to the same 2-cocycle class, and $\beta(x,y)^{|G|} = \alpha(x,y)^{|G|}\mu(x)^{-1}\mu(y)^{-1}\mu(xy)$. However, recall that we have the identity:

$$\mu(x)({}^x\mu(y)) = \mu(xy)\alpha(x,y)^{|G|}.$$

Hence $\beta(x,y)^{|G|} = 1$. Therefore $\beta(x,y)$ takes values in Z for all $x, y \in G$.

(iii) With notation as before, if k has characteristic p, then Z is trivial, and if k is perfect, then every element is a pth power. In particular, for any value $\mu(x) \in k^\times$, there exists an element $\nu(x) \in k^\times$ satisfying $\nu(x)^{|P|} = \mu(x)$. Thus the 2-cocycle β as defined before is in the same cocycle class as 1.

□

Proposition 1.11. Let G be a finite cyclic group and k algebraically closed. Consider $k^\times$ with the trivial action of G. Then $H^2(G; k^\times)$ is trivial.

Proof. Let $G = \langle x \rangle$, $n = |G|$, and $\alpha \in Z^2(G; k^\times)$. Denote by $\hat{x}$ the image of x in the twisted group algebra $k_\alpha G$. Since $x^n = 1$, $\hat{x}^n = \mu 1$ for some $\mu \in k^\times$. Since k is algebraically closed, there exists $\nu \in k^\times$ for which $\nu^n = \mu\alpha(1,1)$. Define $\tilde{x} = \nu^{-1}\hat{x}$. Then, we may compute

$$\tilde{x}^n = \nu^{-n}\hat{x}^n = \mu^{-1}\alpha(1,1)^{-1}\mu 1 = \alpha(1,1)^{-1}1.$$

From 1.5, this is the unit element of $k_\alpha G$. Therefore, the map sending powers of x to the corresponding powers of $\tilde{x}$ induces an algebra isomorphism $kG \cong k_\alpha G$, and thus by 1.6, α represents the trivial class. □

We finish by noting there is a more structural interpretation of $H^2(G; Z)$ (those who are in Marty's reading will recall this!), in terms of invariants of $\mathbb{Z}G$-modules. We define $H^i(G; Z) = \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, Z)$, where Z is viewed as a $\mathbb{Z}G$-module with the prescribed action of G on Z. (The definition of Ext may take a bit too long to define here, but it involves projective resolutions of $\mathbb{Z}$ by $\mathbb{Z}G$-modules.)

2 Group Extensions & Group Cohomology

We now shift gears to now review the parameterization of group extensions in terms of second cohomology groups.

Definition 2.1. Consider the short exact sequence,

$$1 \longrightarrow Z \xrightarrow{\iota} \hat{G} \xrightarrow{\pi} G \longrightarrow 1$$

with Z abelian, that is, ι is injective, π is surjective, and $\ker(\pi) = \mathrm{im}(\iota)$. We say $\hat{G}$ **is an extension of G by Z.** If $\iota(Z)$ is contained in $Z(\hat{G})$, then such an extension is called a **central extension of G by Z**.

In general, an extension of G by an abelian group Z induces an action of G on Z as follows (identifying Z with its image $\iota(Z) \subset \hat{G}$): if $x \in G$ and $a \in Z$, define ${}^x a = \hat{x} a \hat{x}^{-1}$, where $\hat{x} \in \hat{G}$ is any inverse image of x via π. From exactness, any two different inverse images of x in $\hat{G}$ differ by an element in Z, and since Z is abelian, this definition does not depend on the choice of $\hat{x}$. Thus, the action is well-defined. The action is trivial precisely when $\hat{G}$ is a central extension of G by Z.

Remark 2.2. Any extension of G by Z as above defines an element $\alpha \in Z^2(G; Z)$ in the following way. For any $x \in G$, choose $\hat{x} \in \hat{G}$ such that $\pi(\hat{x}) = x$. Then, for any $x, y \in G$, we have that $\pi(\widehat{xy}) = xy = \pi(\hat{x}\hat{y})$. Therefore, $\widehat{xy}$ and $\hat{x}\hat{y}$ differ by a unique element in Z (more precisely, in $\iota(Z)$). Therefore, we may define $\alpha(x, y)$ (with some abuse of notation) to be the unique element of Z satisfying:

$$\hat{x}\hat{y} = \alpha(x, y)\widehat{xy}.$$

By performing the same computation as we did in 1.1, we show that associativity of group multiplication in $\hat{G}$ is equivalent to the 2-cocycle identity of α, and thus conclude $\alpha \in Z^2(G; Z)$.

Caution! As currently written, the 2-cocycle α depends on the choice of elements $\hat{x} \in \pi^{-1}(x)$, but soon, we will show that the class of α in $H^2(G; Z)$ is independent of this choice.

Proposition 2.3. The 2-cocycle α represents the trivial class in $H^2(G; Z)$ if and only if the above *central* extension is split.

Proof. First, suppose the central extension is split, so there exists a section $\sigma : G \to \hat{G}$ such that $\pi \circ \sigma = \mathrm{id}_G$. Then, choosing $\hat{x} = \sigma(x)$ for all $x \in G$ defines the constant 2-cocycle 1.
Now, suppose for some choice of elements $\hat{x}$, $\alpha \in B^2(G; Z)$, that is, there exists a map $\mu : G \to Z$ such that $\alpha(x, y) = \mu(x)({}^x\mu(y))\mu(xy)^{-1}$ for all $x, y \in G$. Define $\sigma(x) = \mu(x)^{-1}\hat{x}$. It is clear that $\pi \circ \sigma = \mathrm{id}_G$, as $\mu(x) \in \mathrm{im}(\iota) = \ker(\pi)$. We show that σ is a homomorphism:

$$\sigma(xy) = \mu(xy)^{-1}\widehat{xy} = \mu(x)^{-1}({}^x\mu(y))^{-1}\hat{x}\hat{y} = \mu(x)^{-1}\mu(y)^{-1}\hat{x}\hat{y} = \sigma(x)\sigma(y).$$

This σ is a section of π, hence the central extension is split. □

Remark 2.4. Conversely to 2.2, any $\alpha \in Z^2(G; Z)$ occurs as a 2-cocycle of an extension of G in Z. As a set, we take $\hat{G} = Z \times G$. We endow $\hat{G}$ with the product defined by

$$(\lambda, x)(\mu, y) = (\alpha(x, y)\lambda\mu, xy),$$

where $x, y \in G$ and $\lambda, \mu \in Z$. We perform the same computations as in 1.5 to show that $\hat{G}$ is a group with unit element $(\alpha(1, 1)^{-1}, 1)$. Define $\pi : \hat{G} \to G$ as the projection $\pi(\lambda, x) = x$ for $(\lambda, x) \in \hat{G}$. This is a surjective group homomorphism with kernel $\{(\lambda, 1) \mid \lambda \in Z\} \cong Z$, hence we have a short exact sequence. Finally by setting $\hat{x} = (1, x)$ for all $x \in G$, we obtain $\hat{x}\hat{y} = (\alpha(x, y), xy) = \alpha(x, y)\widehat{xy}$, and thus, α is determined by this extension.

Sam Side Remark: In A Gentle Course in Local Class Field Theory, by Pierre Guillot, a slightly different construction of $\hat{G}$ is used: here, they endow the group multiplication with a twist:

$$(\lambda, x)(\mu, y) = (\alpha(x, y)\lambda({}^x\mu), xy).$$

Multiple group extensions may correspond to the same 2-coboundaries!

Theorem 2.5. Let G be a group and Z a multiplicatively written abelian group. Let

$$1 \longrightarrow Z \longrightarrow \hat{G} \xrightarrow{\pi} G \longrightarrow 1$$

$$1 \longrightarrow Z \longrightarrow \check{G} \xrightarrow{\tau} G \longrightarrow 1$$

be two extensions of G by Z. Suppose that the two extensions induce the same action of G on Z. For any $x \in G$ choose $\hat{x} \in \hat{G}$ such that $\pi(\hat{x}) = x$ and $\check{x} \in \check{G}$ such that $\tau(\check{x}) = x$. Denote by α, β the 2-cocycles in $Z^2(G; Z)$ satisfying $\hat{x}\hat{y} = \alpha(x, y)\widehat{xy}$ and $\check{x}\check{y} = \beta(x, y)\widecheck{xy}$ for all $x, y \in G$. The classes of α and β in $H^2(G; Z)$ are equal if and only if there exists a group isomorphism $\varphi : \hat{G} \cong \check{G}$ for which the following diagram commutes:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Z & \longrightarrow & \hat{G} & \xrightarrow{\pi} & G & \longrightarrow & 1 \\ & & \downarrow{=} & & \downarrow{\varphi} & & \downarrow{=} & & \\ 1 & \longrightarrow & Z & \longrightarrow & \check{G} & \xrightarrow{\tau} & G & \longrightarrow & 1 \end{array}$$

Proof. Suppose that φ exists which makes the above diagram commute. Then for any $x \in G$, we have $\tau(\varphi(\hat{x})) = \pi(\hat{x}) = x = \tau(\check{x})$. Thus by exactness there is $\gamma(x) \in Z$ such that $\varphi(\hat{x}) = \varphi(x)\check{x}$. It follows that:

$$\varphi(\hat{x}\hat{y}) = \varphi(\alpha(x, y)\widehat{xy}) = \alpha(x, y)\varphi(\widehat{xy}) = \alpha(x, y)\gamma(xy)\widecheck{xy}$$

$$\varphi(\hat{x})\varphi(\hat{y}) = \gamma(x)\check{x}\gamma(y)\check{y} = \gamma(x)\check{x}\gamma(y)\check{x}^{-1}\check{x}\check{y} = \gamma(x)({}^x\gamma(y))\beta(x, y)\widecheck{xy}$$

These two expressions coincide if and only if $\alpha(x, y) = \beta(x, y)\gamma(x)({}^x\gamma(y))\gamma(xy)^{-1}$, in other words, precisely when α and β differ by a coboundary. Conversely, if α and β differ by a coboundary given by γ, we may define φ by the formula $\varphi(z\hat{x}) = \gamma(x)z\check{x}$ for any $x \in G$, $z \in Z$, and verify that it satisfies all the desired properties. □

Before, we noted that the image of $1 \in G$ in $k_\alpha G$ may not be the unit element of the algebra. However, we will now see that one can always achieve this by making a suitable choice for the 2-cocycle α.

Definition 2.6. Let G be a group and Z a multiplicatively written abelian group on which G acts. A 2-cocycle $\alpha \in Z^2(G; Z)$ is called **normalized** if $\alpha(1, x) = 1$ for all $x \in G$.
One may note that in the language used in 2.5, α is normalized if and only if $\hat{1}_G = 1_{\hat{G}}$, that is, the identity of $\hat{G}$ is chosen as the inverse image of the identity of G.

Proposition 2.7. Let G and Z be as in . Any class in $H^2(G;Z)$ can be represented by a normalized 2-cocycle.

Proof. Let $\alpha \in Z^2(G;Z)$. By 1.3, $\alpha(1,1) = \alpha(1,x)$ for all $x \in G$. Set $\mu(x) = \alpha(1,1)$, then define

$$\beta(x,y) = \alpha(x,y)\mu(x)^{-1}({}^x\mu(y))^{-1}\mu(xy) = \alpha(x,y)({}^x\alpha(1,1)^{-1}).$$

$\beta \in Z^2(G;Z)$ represents the same class as α by definition and $\beta(1,x) = \beta(1,1) = 1$ by construction. □

We now return our focus to twisted group algebras. Let $\alpha \in H^2(G;k^\times)$ and let $H \leqslant G$. Suppose that $res^G_H(\alpha) = 1 \in H^2(H;k^\times)$, then by 1.6, the inclusion $\iota : H \to G$ and some map $\gamma : H \to k^\times$ yield an injective algebra homomorphism $kH \to k_\alpha G$ sending $y \in H$ to $\gamma(y)y \in k_\alpha G$. In this way, $k_\alpha G$ becomes a $kH-kH$-bimodule, with kH as a direct summand in its decomposition. The next proposition concerns this scenario.

Proposition 2.8. Let G be a group, H a subgroup of finite index in G, and Z an abelian group, written multiplicatively, on which G acts. Let $\alpha \in H^2(G;Z)$. If $\mathrm{res}^G_H(\alpha)$ is the trivial class in $H^2(H;Z)$, then $\alpha^{[G:H]}$ is the trivial class in $H^2(G;Z)$.

Proof. Consider a group extension

$$1 \longrightarrow Z \xrightarrow{\ \iota\ } \hat{G} \xrightarrow{\ \pi\ } G \longrightarrow 1$$

and choose elements $\hat{x} \in \hat{G}$ such that $\pi(\hat{x}) = x$ for $x \in G$ and such that $\hat{x}\hat{y} = \alpha(x,y)\widehat{xy}$, where the book again abusively denotes by α a 2-cocycle representing the class α. By assumption, α restricts to an element in $B^2(H;Z)$, so we may assume that α is constantly 1 on $H \times H$. Therefore, given $y, z \in H$, $\hat{y}\hat{z} = \widehat{yz}$.

Denote by $\mathcal{R}$ a set of coset representatives of G/H in G. Then, ever element in G can be written uniquely as xh for some $x \in \mathcal{R}$ and $h \in H$. We now modify our choice of $\hat{x}$: if $x \in \mathcal{R}$ and $h \in H$, we keep our previous choices of $\hat{x}$ and $\hat{h}$ and set $\widehat{xh} := \hat{x}\hat{h}$ for all other elements of G. (Recall we may do this via 2.5, as we are working within a cocycle class.) From this choice, we may verify that $\hat{x}\hat{h} = \widehat{xh}$ for any $x \in G, h \in H$. Thus $\alpha(x,h) = 1$ when $h \in H$. The 2-cocycle identity

$$\alpha(x,yh)({}^x\alpha(y,h)) = \alpha(xy,h)\alpha(x,y)$$

implies that $\alpha(x,yh) = \alpha(x,y)$ for any $x, y \in G$ and $h \in H$. In other words $\alpha(x,y)$ depends only on the H-coset of y for the 2nd variable. Thus, the expression

$$\mu(x) = \prod_{y \in \mathcal{R}} \alpha(x,y)$$

does not depend on the choice of $\mathcal{R}$. Now, for $x, y, z \in G$ consider the 2-cocycle identity:

$$\prod_{z \in \mathcal{R}} \alpha(x,yz)({}^x\alpha(y,z)) = \alpha(xy,z)\alpha(x,y).$$

This gives:

$$\mu(x)({}^x\mu(y)) = \mu(xy)\alpha(x,y)^{[G:H]},$$

and hence $\alpha^{[G:H]} \in B^2(G;Z)$, as desired. □

Corollary 2.9. Let G be a finite group, p prime, P a Sylow p-subgroup of G, and Z an abelian p-group on which G acts. The restriction map $\mathrm{res}^G_P : H^2(G;Z) \to H^2(P;Z)$ is injective.

Proof. Let $m = [G : P]$, a positive integer which p does not divide. If $\alpha \in H^2(G;Z)$ satisfies $\mathrm{res}^G_P(\alpha) = 0$, then $\alpha^m = 0$ by 2.8, and thus $\alpha = 0$ since taking mth powers is an automorphism of Z, since Z is an abelian p-group. □

Note that a twisted group algebra $k_\alpha G$ does not in general have a basis which is closed under multiplication, but the next proposition shows that it is a quotient of the group algebra $k\hat{G}$ of the central extension $\hat{G}$ of G determined by α.

Proposition 2.10. Let Z be an abelian group, written multiplicatively, and let

$$1 \longrightarrow Z \xrightarrow{\iota} \hat{G} \xrightarrow{\pi} G \longrightarrow 1$$

be a central extension of a group G by Z. For any $x \in G$, choose $\hat{x} \in \hat{G}$ such that $\pi(\hat{x}) = x$ and denote by β the 2-cocyle in $Z^2(G;Z)$ determined by $\hat{x}\hat{y} = \beta(x,y)\widehat{xy}$ for $x, y \in G$. Let $\mu : Z \to k^\times$ be a group homomorphism. Then $\alpha = \mu \circ \beta$ is a 2-cocycle in $Z^2(G;k^\times)$ and the map sending $z\hat{x}$ to $\mu(z)x$ for any $x \in G$ and $z \in Z$ induces a surjective algebra homomorphism

$$k\hat{G} \to k_\alpha G$$

Proof. Since μ is a group homomorphism, applying μ to the 2-cocycle identity for β shows that α satisfies the 2-cocycle identity (or use functoriality). For the last statement, call the map φ. Using centrality, we see:

$$\begin{aligned}\varphi(z_1\hat{x_1}) \cdot \varphi(z_2\hat{x_2}) &= \mu(z_1)\mu(z_2)x_1 \cdot x_2 \\ &= \mu(z_1z_2)\alpha(x,y)x_1x_2 \\ &= \mu(z_1z_2\beta(x_1,x_2))x_1x_2 \\ &= \varphi(z_1z_2\beta(x_1,x_2)\widehat{x_1x_2}) \\ &= \varphi(z_1\hat{x_1}z_2\hat{x_2})\end{aligned}$$

□

The book now admits to the confusion of multiplying elements in G and Z together in $\hat{G}$, as for example, elements of $k^\times$ could either be group elements of $\hat{G}$ or scalars of $k\hat{G}$. The book decides to give a name to the inclusion $\iota : k^\times \to \hat{G}$ for expressing elements of $k^\times$ as elements of group extensions. (Are there more things to note here?)

Proposition 2.11. Suppose that k is an algebraically closed field. Let G be a finite group which acts trivially on $k^\times$, let $\alpha \in H^2(G; k^\times)$, and let

$$1 \longrightarrow k^\times \xrightarrow{\iota} \hat{G} \xrightarrow{\pi} G \longrightarrow 1$$

be a central extension of G by $k^\times$ representing α. There is a finite subgroup G' of $\hat{G}$ with the following properties:

(i) We have $\hat{G} = \iota(k^\times) \cdot G'$ and $Z = \iota(k^\times) \cap G'$ is equal to the subgroup of $|G|$th roots of unity in $k^\times$. In particular, $|G'| = |Z| \cdot |G|$ and the exponent of G' divides $|G|^2$.

(ii) The inclusion $G' \to \hat{G}$ induces an isomorphism of k-algebras $kG' \cdot e_Z \cong k_\alpha G$, where e_Z is the idempotent in $Z(kG')$ defined by

$$e_Z = \frac{1}{|Z|} \sum_{z \in Z} z$$

Proof. As usual, denote by $\hat{x}$ an inverse image of x in $\hat{G}$. Then α is represented by the 2-cocycle, abusively (still) denoted α satisfying $\hat{x} \cdot \hat{y} = \alpha(x, y)\widehat{xy}$ for all $x, y \in G$. By 1.10(ii), there exists some map $\mu : G \to k^\times$ for which $\beta(x, y) = \alpha(x, y)\mu(x)\mu(y)\mu(xy)^{-1}$ has values in the subgroup Z of all $|G|$-th roots of unity of $k^\times$. Moreover, we may choose β to be normalized by 2.7.

Now, set $\tilde{x} = \mu(x)\hat{x}$. Then, for $x, y \in G$, we have

$$\tilde{x} \cdot \tilde{y} = \mu(x)\hat{x} \cdot \mu(y)\hat{y} = \mu(x)\mu(y)\alpha(x, y)\widehat{xy} = \beta(x, y)\mu(xy)\widehat{xy} = \beta(x, y)\widetilde{xy}.$$

Since β is normalized we have $\tilde{1}_G = 1_{\tilde{G}}$, thus:

$$G' = \{\zeta\tilde{x} \mid \zeta \in Z, x \in G\} \leqslant \hat{G}$$

satisfies both $\hat{G} = k^\times \cdot G'$ and $Z = k^\times \cap G'$, as desired for (i).

Now, note that $|Z|$ is necessarily invertible in $k^\times$, therefore the element e_Z as defined above is an idempotent in $Z(kG')$ (recalling from the previous chapter). The inclusion from G' to $\hat{G}$ induces an algebra homomorphism $kG' \to k\hat{G}$. In addition, 2.10 gives us a map sending $\tilde{x} = \mu(x)\hat{x}$ to $x \in G$, viewed as an element in $k_\alpha G$ (here, I believe the homomorphism in question for using 2.10 is the trivial one on $k^\times$). Composing these maps yields an algebra homomorphism $kG' \to k_\alpha G$. Since the canonical map $\hat{G} \to G$ sends G' onto G (one may check this from the definition of G'), this algebra homomorphism is surjective, and by construction of G', this homomorphism sends every element $z \in Z$ as viewed as an element of G' to 1. Therefore, it sends e_Z to 1, and thus induces a surjective algebra homomorphism $kG' \cdot e_Z \to k_\alpha G$. However, $|G'|/|Z| = |G|$, thus both algebras have the same dimension, and hence are isomorphic. □