# 1.4: Category Algebras & Cohomology in $\mathcal{C}$

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## 1 Category Algebras

Category algebras can be seen as a straightforward generalization of group algebras and monoid algebras. Given a category $\mathcal{C}$ and a unital, commutative ring $k$, we wish to build a $k$-module out of the data of $\mathcal{C}$.

We say a category $\mathcal{C}$ is **small** if its class of objects, $\mathrm{Ob}(\mathcal{C})$, and class of morphisms, $\mathrm{Mor}(\mathcal{C})$, both form sets, and $\mathcal{C}$ is **finite** if both sets are finite. Note that if $\mathrm{Mor}(\mathcal{C})$ forms a set (resp. is finite), then $\mathrm{Ob}(\mathcal{C})$ necessarily forms a set (resp. is finite) as well. We say $\mathcal{C}$ is **large** if it is not small, that is, $\mathrm{Mor}(\mathcal{C})$ does not form a set. One such example of a large category is **Set**, due to Russell's paradox.

**Definition 1.1.** Let $\mathcal{C}$ be a small category. **The category algebra $k\mathcal{C}$ of $\mathcal{C}$ over $k$** is the free $k$-module with basis $\mathrm{Mor}(\mathcal{C})$, endowed with the unique $k$-bilinear multiplication defined for all $\varphi, \psi \in \mathrm{Mor}(\mathcal{C})$ by $\varphi \cdot \psi = \varphi \circ \psi$ whenever the composition of morphisms is defined, and $\varphi \cdot \psi = 0$ otherwise.

Equivalently, $k\mathcal{C}$ consists of all functions $f : \mathrm{Mor}(\mathcal{C}) \to k$ that are zero on all but finitely many morphisms. Given two such functions $f, g$, the product $g \cdot f$ in $k\mathcal{C}$ is the function sending $\alpha \in \mathrm{Mor}(\mathcal{C})$ as follows:

$$\alpha \mapsto \sum_{\varphi \circ \psi = \alpha} g(\varphi) f(\psi),$$

The sum is finite by assumption, and is assumed to be zero if the set of indexing pairs is empty.

**A Quick Explanation:** In this equivalence of definitions, the function that corresponds to $\varphi \in \mathrm{Mor}(\mathcal{C})$ considered as an element of $k\mathcal{C}$ is as follows:

$$f_\varphi(\alpha) = \begin{cases} 1 & \alpha = \varphi \\ 0 & \alpha \neq \varphi \end{cases}$$

Let's verify multiplication works as we hope it would. Let $\varphi, \psi \in \mathrm{Mor}(\mathcal{C})$. If their composition is well-defined, then $\varphi \cdot \psi = \varphi \circ \psi \in k\mathcal{C}$. Otherwise, $\varphi \cdot \psi = 0$. In the alternative definition, we have that the corresponding product $f_\varphi \cdot f_\psi$ is given by:

$$h : \alpha \mapsto \sum_{\beta \circ \gamma = \alpha} f_\varphi(\beta) \circ f_\psi(\gamma).$$

The only nonzero term in this sum happens precisely when $\beta = \varphi$ and $\gamma = \psi$ (if possible), which only occurs when $\alpha = \varphi \circ \psi$, thus $h = f_{\varphi \circ \psi}$ when the composition is possible, as desired. □

We note that for any $X \in \mathrm{Ob}(\mathcal{C})$, the identity morphism $\mathrm{Id}(X)$ is an idempotent in $k\mathcal{C}$ and for any two unique objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, $\mathrm{id}(X)$ and $\mathrm{id}(Y)$ are orthogonal. $k\mathcal{C}$ need not be unital in general, but if $\mathcal{C}$ is finite, then $k\mathcal{C}$ is a unitary algebra with unit element

$$1_{k\mathcal{C}} = \sum_{X \in \mathrm{Ob}(\mathcal{C})} \mathrm{id}_X \,.$$

We next introduce a few examples of category algebras.

**Example 1.2.** Group algebras, monoid algebras, and matrix algebras are all examples of category algebras with the right choice of category.

(a) Let $G$ be a group or monoid. Denote by $\mathbf{G}$ the one-object category whose endomorphisms are in bijection with the elements of $G$, and such that composition of endomorphisms is equivalent to multiplication in $G$. The map sending $x \in G$ to $x \in \mathrm{Mor}(\mathbf{G})$ induces an isomorphism of $k$-algebras $kG \cong k\mathbf{G}$.

(b) Let $n \in \mathbb{N}^+$ and denote by $\mathbf{M}_n$ a category with $n$ objects, $X_1, \ldots, X_n$ with a unique morphism $\varphi_{i,j} : X_i \to X_j$ for each pair $(i, j)$ satisfying $1 \leqslant i, j \leqslant n$ (so in total, there are $n^2$ morphisms). Let $E_{i,j} \in \mathrm{Mat}_n(k)$ be the matrix with a 1 in the $i, j$-th entry, and 0s elsewhere. Then, the map sending $\phi_{i,j}$ to $E_{i,j}$ induces an isomorphism of $k$-algebras $k\mathbf{M}_n \cong \mathrm{Mat}_n(k)$.

(c) Let $(\mathcal{P}, \leqslant)$ be a partially ordered set (referred to in short by *poset*). $\mathcal{P}$ can be viewed as a category with a unique morphism $x \to y$ for all $x, y \in \mathcal{P}$ for which $x \leqslant y$. The corresponding category algebra $k\mathcal{P}$ is called the **incidence algebra of** $\mathcal{P}$. If $\mathcal{P}$ is a finite poset, note that $k\mathcal{P}$ is a subalgebra of $k\mathbf{M}_n$ and thus isomorphic to a subalgebra of $\mathrm{Mat}_n(k)$. For example, Exercise 1.4.14 notes that the category algebra arising from the totally ordered set $\{1, 2, \ldots, n\}$ is isomorphic to the algebra of upper triangular matrices.

**Remark 1.3.** Every $k$-algebra $A$ can be obtained as a quotient of a monoid algebra. We do so as follows: take the multiplicative monoid $(A, \cdot)$ with a map $kA \to A$ given by evaluation (the book calls this the "obvious map," and it is indeed obvious if you try writing an element of $kA$). There is a similar notion for category algebras; for a finite-dimensional algebra $A$ over an algebraically closed field $k$, there is a canonical choice of category $\mathcal{C}$ with finitely many objects such that $A \cong k\mathcal{C}/I$ for some ideal $I$. The book notes that $I$ in general cannot be chosen canonically, but this leads to the notion of a **quiver** of an algebra, see chapter 4.9 for more details.

A category algebra is never "far off" from a semigroup algebra (an algebra obtained from a semigroup). We perform the construction is as follows: given a category $\mathcal{C}$ with at least two objects, define a semigroup $\mathcal{S}$ by adjoining a zero element. Precisely, set $\mathcal{S} = \mathrm{Mor}(\mathcal{C}) \cup \{\nu\}$ with $\nu \cdot \varphi = \nu = \varphi \cdot \nu$ for all $\varphi \in \mathcal{S}$. For $\varphi, \psi \in \mathrm{Mor}(\mathcal{C})$, $\varphi \cdot \psi = \varphi \circ \psi$ if the composition is defined, and $\varphi \cdot \psi = \nu$ otherwise. Then $k\nu$ is an ideal in $k\mathcal{S}$ and there is an isomorphism $k\mathcal{C} \cong k\mathcal{S}/k\nu$.

**Recall:** Given two categories $\mathcal{C}$ and $\mathcal{D}$, an **equivalence of categories** $\mathcal{C}$, $\mathcal{D}$ consists of a covariant functor $F : \mathcal{C} \to \mathcal{D}$ which is:

- *Full*: For all $X_1, X_2 \in \mathrm{Ob}(\mathcal{C})$, the map induced by $\mathrm{Hom}_{\mathcal{C}}(X_1, X_2) \to \mathrm{Hom}_{\mathcal{D}}(F(X_1), F(X_2))$ is surjective.
- *Faithful*: For all $X_1, X_2 \in \mathrm{Ob}(\mathcal{C})$, the map induced by $\mathrm{Hom}_{\mathcal{C}}(X_1, X_2) \to \mathrm{Hom}_{\mathcal{D}}(F(X_1), F(X_2))$ is injective.
- *Essentially Surjective*: For all $Y \in \mathrm{Ob}(\mathcal{D})$, there exists some $X \in \mathrm{Ob}(\mathcal{C})$ for which $F(X) \cong Y$.

Equivalently, $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if there exists some covariant functor $G : \mathcal{D} \to \mathcal{C}$ for which $FG \cong 1_{\mathcal{D}}$ and $GF \cong 1_{\mathcal{C}}$.

A covariant functor between small categories $\Phi : \mathcal{C} \to \mathcal{D}$ induces a unique $k$-linear map $\varphi : k\mathcal{C} \to k\mathcal{D}$. Take caution, however! This map need not be an algebra homomorphism! For example, using 1.2, note that all objects in $\mathbf{M}_n$ are isomorphic and hence there is a unique equivalence of categories $\mathbf{M}_n \to \mathbf{M}_1$, sending each object of $\mathbf{M}_n$ to the unique object of $\mathbf{M}_1$ and all morphisms to the unique morphism (more generally, any category is equivalent to its skeleton). This induces a $k$-linear map $\mathrm{Mat}_n(k) \to \mathrm{Mat}_1(k) \cong k$, which sums the elements of the $n \times n$ matrix, but this map is clearly not multiplicative unless $n = 1$.

**Proposition 1.4.** Let $\mathcal{C}, \mathcal{D}$ be small categories and $\Phi : \mathcal{C} \to \mathcal{D}$ a covariant functor. Let $\alpha : k\mathcal{C} \to k\mathcal{D}$ be the unique $k$-linear map induced by $\Phi$. Then $\alpha$ is a $k$-algebra homomorphism if and only if the object map induced by $\Phi$, $\mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ is injective. Moreover, if $\mathrm{Ob}(\mathcal{C}), \mathrm{Ob}(\mathcal{D})$ are finite, then $\alpha$ is a *unital* $k$-algebra homomorphism if and only if $\Phi$ induces a bijection $\mathrm{Ob}(\mathcal{C}) \cong \mathrm{Ob}(\mathcal{D})$.

*Proof.* First, suppose $\Phi$ is not injective on objects. Let $X, X'$ be two different objects in $\mathcal{C}$ such that $\Phi(X) = \Phi(X') = Y \in \mathrm{Ob}(\mathcal{D})$. Then $\mathrm{id}_X \cdot \mathrm{id}_{X'} = 0$ in $k\mathcal{C}$, so $\Phi(\mathrm{id}_X \cdot \mathrm{id}_{X'}) = 0$ in $k\mathcal{D}$. However, $\Phi(\mathrm{id}_X) \cdot \Phi(\mathrm{id}_{X'}) = \mathrm{id}_Y \cdot \mathrm{id}_Y = \mathrm{id}_Y \neq 0$, and thus, $\alpha$ is not multiplicative. Conversely, suppose $\Phi$ is injective. Then for all $\varphi, \psi \in \mathrm{Mor}(\mathcal{C})$ such that $\Phi(\varphi), \Phi(\psi)$ are composable in

$\mathcal{D}$, the morphisms $\varphi, \psi$ are composable as well by injectivity. Thus, $\alpha(\varphi \cdot \psi) = \Phi(\varphi \circ \psi)$, hence $\alpha$ is multiplicative.
Finally, the last statement regarding unitality of $\alpha$ can be observed from the fact that the unit elements of $k\mathcal{C}$ and $k\mathcal{D}$ are equal to $\sum_{X \in \mathrm{Ob}(\mathcal{C})} \mathrm{id}_X$ and $\sum_{Y \in \mathrm{Ob}(\mathcal{D})} \mathrm{id}_Y$ respectively. □

We wrap up this first section by drawing an equivalence between category algebras $k\mathcal{C}$ and functors from $\mathcal{C}$ to the category of $k$-modules, $k\mathbf{Mod}$, when $\mathrm{Ob}(\mathcal{C})$ is finite. (We deviate from the textbook notation a bit because I feel like it.)

**Recall:** Denote by $\mathcal{D}^{\mathcal{C}}$ the **functor category**, whose objects are functors $\mathcal{C} \to \mathcal{D}$, and whose morphisms are given by natural transformations. Composition is defined by vertical composition of transformations, see Riehl's *Category Theory In Context*, chapter 1.7 for a more complete discussion. We note if $\mathcal{C}$ and $\mathcal{D}$ are small (resp. finite), then so is $\mathcal{D}^{\mathcal{C}}$.
A **$k$-linear category** is a category $\mathcal{C}$ whose morphism sets have a $k$-module structure such that composition is $k$-bilinear. A **$k$-linear functor** is a functor between $k$-linear categories which is $k$-linear on morphism sets.

**Theorem 1.5.** Let $\mathcal{C}$ be a small category with finite object set. There is a $k$-linear equivalence $k\mathcal{C}\mathbf{Mod} \cong k\mathbf{Mod}^{\mathcal{C}}$.

The proof supplied by the book skips over some key details in the proof, namely, all the details about the morphism maps of the constructed functors. We will fill in the blanks regarding the morphisms, but not perform a complete proof as those details end up being rather lengthy.

*Proof.* Let $M$ be a left $k\mathcal{C}$-module. We first construct the forward direction of the equivalence by defining a functor $F_M : \mathcal{C} \to k\mathbf{Mod}$ as follows: for each object $X$ in $\mathcal{C}$, set $F_M(X) := \mathrm{id}_X \cdot M$. For each morphism $\varphi : X \to Y$ in $\mathcal{C}$ note that $\varphi = \mathrm{id}_Y \circ \varphi = \varphi \circ \mathrm{id}_X$. Therefore, for any $\mathrm{id}_X \cdot m \in \mathrm{id}_X \cdot M$,

$$\varphi \cdot (\mathrm{id}_X \cdot m) = (\varphi \circ \mathrm{id}_X) \cdot m = (\mathrm{id}_Y \circ \varphi) \cdot m = \mathrm{id}_Y \cdot (\varphi \cdot m) \in \mathrm{id}_Y \cdot M,$$

Hence, the action of $\varphi$ on $M$ induces a unique $k$-linear map $F_M(\varphi) : \mathrm{id}_X \cdot M \to \mathrm{id}_Y \cdot M$, and through this assignment, $F_M$ is a functor. Note that $F$ is contravariant, as the order of composition is reversed under this assignment. Call this assignment $\rho$.
(Sketch) $\rho$ maps $k\mathcal{C}$-module homomorphisms to natural transformations of functors $\mathcal{C} \to k\mathbf{Mod}$ as follows: given two $k\mathcal{C}$-modules $M, N$ and a module homomorphism $f : M \to M'$, we define the natural transformation $\rho(f) : F_M \to F_N$ by:

$$\rho(f)_X : F_M(X) \to F_N(X), \quad \mathrm{id}_X \cdot m \to \mathrm{id}_X \cdot f(m)$$

It follows immediately that by $k\mathcal{C}$-linearity of $f$, we obtain the commutative diagram:

$$\begin{array}{ccc} F_M(X) & \xrightarrow{\rho(f)_X} & F_N(X) \\ \downarrow{\scriptstyle F_M(\varphi)} & & \downarrow{\scriptstyle F_N(\varphi)} \\ F_M(Y) & \xrightarrow{\rho(f)_Y} & F_N(Y) \end{array}$$

Hence $\rho(f)$ is indeed a natural transformation. With some work, we may verify that the assignment is functorial with respect to $\rho$, and that the map is $k$-linear (details omitted). Thus, the assignment $M \mapsto F_M$ defines a $k$-linear functor $\rho : k\mathcal{C}\mathbf{Mod} \to k\mathbf{Mod}^{\mathcal{C}}$.

Conversely, given some functor $F : \mathcal{C} \to k\mathbf{Mod}$, we define a $k\mathcal{C}$-module $M_F$ by setting

$$M_F = \bigoplus_{X \in \mathrm{Ob}(\mathcal{C})} F(X),$$

and for any morphism $\varphi : X \to Y$ in $\mathcal{C}$ and $m \in F(Z)$, we define the $k\mathcal{C}$-module structure on $M_F$ by setting $\varphi \cdot m = (F(\varphi))(m)$ if $Z = X$ and $\varphi \cdot m = 0$ if $Z \neq X$. Call this assignment $\sigma$

(Sketch) $\sigma$ maps natural transformations of functors $\mathcal{C} \to k\mathbf{Mod}$ to $k\mathcal{C}$-module homomorphisms as follows: given two functors $F, G : \mathcal{C} \to k\mathbf{Mod}$, and a natural transformation $\alpha : F \to G$ with corresponding morphisms $\alpha_X : F(X) \to G(X)$, we define a $k\mathcal{C}$-module homomorphism $\sigma(\alpha) : M_F \to M_G$ by:

$$\sigma(\alpha) : M_F \to M_G, \qquad \sum_{X \in \mathrm{Ob}(\mathcal{C})} x_{\mathcal{C}} \mapsto \sum_{X \in \mathrm{Ob}(\mathcal{C})} \alpha_X(x_{\mathcal{C}})$$

With some work, we verify that this is indeed a module homomorphism and $k$-linear (details omitted). Thus, the assignment $F \mapsto M_F$ defines a $k$-linear functor $\sigma : k\mathbf{Mod}^{\mathcal{C}} \to k\mathcal{C}\mathbf{Mod}$ (this requires verification).

Now, we verify these functors witness an equivalence of categories; in fact, they are inverse. We will show this for objects only here. We first verify that $\rho \circ \sigma$ is the identity functor on $k\mathbf{Mod}^{\mathcal{C}}$. We have:

$$\rho \circ \sigma(F)(X) = F_{M_F}(X) = \mathrm{id}_X \cdot \bigoplus_{X \in \mathrm{Ob}(\mathcal{C})} F(X) = \bigoplus_{X \in \mathrm{Ob}(\mathcal{C})} F(\mathrm{id}_X)(F(X)) = F(X)$$

Hence $\rho \circ \sigma$ is the identity functor. Next, we recall that $\mathrm{id}_X$ is an idempotent and $1_{k\mathcal{C}} = \sum_{X \in \mathrm{Ob}(\mathcal{C})} \mathrm{id}_X$. Therefore,

$$M = \bigoplus_{X \in \mathrm{Ob}(\mathcal{C})} \mathrm{id}_X \cdot M.$$

We now verify $\sigma \circ \rho$ is the identity functor on $k\mathcal{C}\mathbf{Mod}$:

$$\sigma \circ \rho(M) = M_{F_M} = \bigoplus_{X \in \mathrm{Ob}(\mathcal{C})} \mathrm{id}_X \cdot M = M.$$

We omit the proof that the composition of $\sigma, \rho$ form the identity functors on morphisms. Thus, $\sigma$ and $\rho$ form an equivalence of categories, as desired. □

# 2 Twisted Category Algebras & Functor 2-Cohomology

There is a twisted version of category algebras that is analogous to twisted group algebras. The following definition is a particular case of low degree functor cohomology; when specialized to the groupoid $\mathbf{G}$ of a group $G$ this yields the corresponding concepts studied in chapter 1.2.

**Definition 2.1.** Let $\mathcal{C}$ be a small category and $M$ an abelian group, written multiplicatively. A **2-cocycle of $\mathcal{C}$ with coefficients in** $M$ is a map $\alpha$ sending any two morphisms $\varphi, \psi$ in $\mathcal{C}$ for which the composition $\psi \circ \varphi$ is defined to an element $\alpha(\psi, \varphi) \in M$ such that for any three morphisms $\varphi, \psi, \tau$ in $\mathcal{C}$ for which the compositions $\psi \circ \varphi$ and $\tau \circ \psi$ are defined we have the **2-cocycle identity**

$$\alpha(\tau, \psi \circ \varphi)\alpha(\psi, \varphi) = \alpha(\tau \circ \psi, \varphi)\alpha(\tau, \psi)$$

in $M$. The set of 2-cocycles of $\mathcal{C}$ with coefficients in $M$ is denoted by $Z^2(\mathcal{C}; M)$, an abelian group with product induced by $M$.

A **2-coboundary of $\mathcal{C}$ with coefficients in** $M$ is a map $\beta$ sending any two morphisms $\varphi, \psi \in \mathcal{C}$ for which $\psi \circ \varphi$ is defined to an element $\beta(\psi, \varphi) \in M$ such that there exists a map $\gamma : \mathrm{Mor}(\mathcal{C} \to M)$ satisfying

$$\beta(\psi, \varphi) = \gamma(\psi)\gamma(\varphi)\gamma(\psi \circ \varphi)^{-1}.$$

The set $B^2(\mathcal{C}; M)$ is a subgroup of $Z^2(\mathcal{C}; M)$ and the quotient group

$$H^2(\mathcal{C}; M) = Z^2(\mathcal{C}; M)/B^2(\mathcal{C}; M)$$

is the **second cohomology group of $\mathcal{C}$ with coefficients in** $M$.

**Definition 2.2.** Let $\mathcal{C}$ be a small category and $\alpha \in Z^2(\mathcal{C}; k^\times)$. The **twisted category algebra of $\mathcal{C}$ by** $\alpha$ is the $k$-algebra, denoted $k_\alpha\mathcal{C}$, which is equal to $k\mathcal{C}$ as a $k$-module, endowed with the unique $k$-bilinear product $k_\alpha\mathcal{C} \times k_\alpha\mathcal{C} \to k_\alpha\mathcal{C}$ defined by $\psi\varphi = \alpha(\psi, \varphi)\psi \circ \varphi$ if the composition $\psi \circ \varphi$ is defined, and $\psi\varphi = 0$ otherwise.

**Theorem 2.3.** Let $\mathcal{C}$ be a small category and $\alpha, \beta \in Z^2(\mathcal{C}; k^\times)$. The following hold:

1. The $k$-algebra $k_\alpha\mathcal{C}$ is associative.

2. For any morphism $\varphi : X \to Y$ in $\mathcal{C}$, we have $\alpha(\mathrm{id}_Y, \mathrm{id}_Y) = \alpha(\mathrm{id}_Y, \varphi) = \alpha(\varphi, \mathrm{id}_X) = \alpha(\mathrm{id}_X, \mathrm{id}_X)$.

3. There is an isomorphism $k_\alpha\mathcal{C} \cong k_\beta\mathcal{C}$ mapping a morphism $\varphi \in \mathcal{C}$ to $\gamma(\varphi)\varphi$ for some map $\gamma : \mathrm{Mor}(\mathcal{C}) \to k^\times$ if and only if the images of $\alpha$ and $\beta$ in $H^2(\mathcal{C}; k^\times)$ are equal.

*Proof.* The proof follows essentially the same as the proofs of the analogous theorems in 1.2. □

Like in the case of groups, second cohomology groups of categories are related to extensions of categories - see the references provided by Linckelmann.

**Remark 2.4.** If $\mathcal{C}$ is such a category such that the sets $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ already have a $k$-module structure and such that composition in $\mathcal{C}$ is bilinear, then there is no need to linearize the homomorphism sets again; one can define an algebra by taking the direct sum $\bigoplus_{X,Y\in \mathrm{Ob}(\mathcal{C})} \mathrm{Hom}_{\mathcal{C}}(X,Y)$ with product induced by composition of morphisms. Theorem 1.4.4 applies in this situation if $\mathrm{Ob}(\mathcal{C})$ is finite: a $k$-linear functor $F : \mathcal{C} \to k\mathbf{mod}$ defines a module $\bigoplus_{X\in \mathrm{Ob}(\mathcal{C})} F(X)$.

# 3 Exercises

**Exercise 3.1.** Let $k$ be a field and $\mathcal{C}$ a finite category. Show that if $\mathcal{C}$ has at least two morphisms, then $k\mathcal{C}$ is not a division ring; that is, $k\mathcal{C}$ has a nonzero noninvertible element.

*Proof.* In the case where $\mathcal{C}$ has more than 1 object, for any object $X$ consider the element $1_X \in k\mathcal{C}$. There does not exist any $f \in \mathrm{Mor}(\mathcal{C})$ such that $1_X \cdot f = 1_Y$ or $f \cdot 1_X = 1_Y$, for any $Y \neq X$. Then by linear independence of morphisms, for all $\alpha \in k\mathcal{C}$, $\alpha \cdot 1_X$ and $1_X \cdot \alpha$ cannot have nonzero $1_Y$ coefficient in its expansion; hence $\alpha$ cannot be the unit of $k\mathcal{C}$. Thus, $f$ is noninvertible in $k\mathcal{C}$.

Otherwise, if $\mathcal{C}$ has 1 object but is not a groupoid, then there exists a non-invertible $f \in \mathrm{Mor}(\mathcal{C})$, i.e., for all $g \neq f$, $g \cdot f \neq 1_X \neq f \cdot g$. Then by linear independence of morphisms, for all $\alpha \in k\mathcal{C}$, $\alpha \cdot f$ and $f \cdot \alpha$ cannot have nonzero $1_X$ coefficient in its expansion; hence $\alpha$ cannot be the unit of $k\mathcal{C}$.

Otherwise, $\mathcal{C}$ is a 1-object groupoid, hence a group. Then consider the element $\varphi \sum_{f\in \mathrm{Mor}(\mathcal{C})} f \in k\mathcal{C}$. We observe that for any $g \in \mathrm{Mor}(\mathcal{C})$, $g \cdot \varphi = \varphi \cdot g = \varphi$. Therefore, for any element of $k\mathcal{C}$, written as a $k$-linear combination $\sum_{f\in \mathrm{Mor}(\mathcal{C})} a_f f$, we have

$$\left(\sum_{f\in \mathrm{Mor}(\mathcal{C})} a_f f\right)\varphi = \left(\sum_{f\in \mathrm{Mor}(\mathcal{C})} a_f\right)\varphi$$

Since $\varphi$ has multiple nonzero coefficients in its expansion, but the unit element only has one, $\varphi$ cannot be invertible. (Note that this construction does not necessarily yield a noninvertible element for non-groupoids, a 2-element monoid yields a counterexample.) □