Finding Joy With Combinatorial Games

Sam K. Miller

February 25, 2021

Introduction

(Though long personal anecdotes are not the norm for mathematical seminars, I was told I could talk about "whatever I want" for an hour, and I thought this introduction may give others food for thought, and perhaps inspiration for why *they* choose to study mathematics.)

In my former professor Francis Su's retiring-presidential-address-turned-blog-post-turned-book "Mathematics For Human Flourishing," he posits the following question to us: "Why do mathematics?" It is strange - why are so many people from differing cultures and backgrounds, which hold different values and beliefs, united to do math, to study this abstract field which demands so much of us? Especially in today's world, where there are so many people who actively degrade math, who wish for it to be removed from standard curricula, thinking it to be a waste of time and taxes? He believe this question is vital for any mathematician or math enthusiast. In his words:

> "Why do mathematics? This is a simple question, but worth considerable reflection. Because how you answer will strongly determine who you think should be doing mathematics, and how you will teach it.
>
> Why is Christopher [an incarcerated student in contact with Francis] sitting in a prison cell studying calculus, even though he won't be using it as a free man for another 25 years? Why was Simone [Weil, philosopher and brother of Andre Veil,] so captivated by transcendent mathematical truths? Why should anyone persist in doing math or seeing herself as a mathematical person when others are telling her in subtle and not so subtle ways she doesn't belong?"

I too believe this is a question worth asking. I fell madly in love with math from a young age, attending math circles, camps, and contests whenever I could through middle and high school. The thought that I could prove, without a doubt, that some property of the numbers would always be satisfied, gave me power. But through my time as an undergrad, I started to lose that special feeling math gave me, instead feeling like I was "going through the motions." I hadn't burned out persay (though the first half of my undergrad could be viewed as residual burnout from high school, where I strove to overachieve in all the ways I possibly could without any regard for sleep or depression), but I was moving forward without any purpose. I graduated a math major, but rather than continuing my studies, decided to become a software engineer and data scientist for the time being. After all, the grass is always greener, and one may lead a lucrative life developing software at large tech companies. It wasn't until I began that job until the feeling of loss began to sat in, and I began to question what about mathematics it was that gave me this feeling of vigor that coding lacked? After all, it's tough to appreciate what you have until it's gone.

Francis's belief is that the practice, not just the pure study, of mathematics allows people to *flourish*, to live a virtuous life by satisfying basic human desires. These desires, in short, are **"play", "beauty", "truth", "justice", and "love."** I too ascribe to this belief. Fast forward to the present day, we are students in graduate school, working rigorously and tirelessly into the night to learn abstract mathematics at the forefront of research so we too may leave our mark in the tome of discovered (or created?) mathematical knowledge. We find **beauty** in learning new powerful results, and **truth** in practicing rigorous mathematical thinking day after day. This pursuit can be tiring, draining, sometimes leaving us (or at least me) with imposter syndrome. However, at UCSC, we find **love** in this community, as we support and look out for each other, creating a sense of belonging, giving all of us the opportunity to flourish. And we are fortunate to find **justice** in a community of graduates who are willing to fight for it, for inclusion and equity within the mathematics community. And for our graduate student body, we fight for a living wage, for workers rights, and for justice for those of us who were wronged by the administration and law enforcement.

However, I have not yet touched upon the final of these desires, **play**. Sometimes, I feel that in graduate school, my time for mathematical play is diminished or even forgotten, and this can again lead to my own questioning, self-doubt. We are loaded up with so many exciting new things to learn and study, but courses move so fast and the weeks fly by, and it becomes difficult for me to stop and smell the flowers, to kick back and relax and appreciate just what we've been learning with a clear, unburdened mind. Again, quoting Francis,

> Mathematics makes the mind its playground. We play with patterns, and within the structure of certain axioms, we exercise freedom in exploring their consequences, joyful at any truths we find. [...]
>
> And mathematical play builds virtues that enable us to flourish in every area of our lives. For instance, math play builds **hopefulness**: when you sit with a puzzle long enough you are exercising hope that you will eventually solve it. Math play builds **community** — when you share in the delight of working on a problem with another human being. And math play builds **perseverance** — just as weekly soccer practices build up the muscles that make us stronger for the next game, weekly math investigations make us more fit for the next problem whatever that is, even if we don't solve the current problem.
>
> Play is part of human flourishing. *You cannot flourish without play.*

To rediscover my own personal play, I looked back to where I found play in the first place. And as I thought, I realized that from the very start, play, and the freedom to explore, is what first drew me to mathematics! Math competitions of course count, but also simple puzzles and games, playful problems. And I've seen the echoes of this feeling, in my Representation Theory or Modular Forms courses, where I once again felt inspired to explore. However earlier on, I predominantly found play in combinatorics. I witnessed beautiful combinatorial arguments in math camps through high school (one of which was hosted at UCSC!), but even before that, perhaps my first exposure to this world was through a box set of DVD lectures by Arthur T. Benjamin named "The Joy of Mathematics." It is quite a happy coincidence that some years later, I would study under him and publish my first research paper together with him!

So in the spirit of mathematical play, and in the spirit of those DVD lectures which lit the spark inside of me, today I will share with you a potpourri of combinatorial games and counting arguments, most of which I originally learned from Arthur, and some original results I proved prior to entering UCSC. Unlike most seminars which I find can require intense focus and a good deal of prerequisite knowledge to follow, my hope is that this talk can remain light and playful, while still eliciting that mathematical beauty and truth that we all search for. These are the types of problems that satisfy my desire of play in mathematics, and I hope I can spread that feeling of play within you as well.

1 The Challenging Knight's Tour

Within the community of Chess, there exists a problem known as *The Knight's Tour*. The most simple variant of the puzzle is as follows: we are given an empty chessboard, and a knight. Recall that a knight moves in an "L"-shape, that is, it moves 2 squares vertically then 1 square horizontally, or vice versa; this rule allows for at most 8 possible moves. The most simple objective of the knight's tour is to place the knight on the board, then perform a series of moves with the knight so that it reaches each square on the board exactly once. We call this an open knight's tour. There are more difficult variants which may be performed:

- The **Closed** Knight's Tour: Perform an open knight's tour, with the addition that the knight ends on the same square that it started on.
- The **Challenging** Knight's Tour: First, choose two designated squares, then place the knight on one of those squares. Perform an open knight's tour which ends on the other designated square.

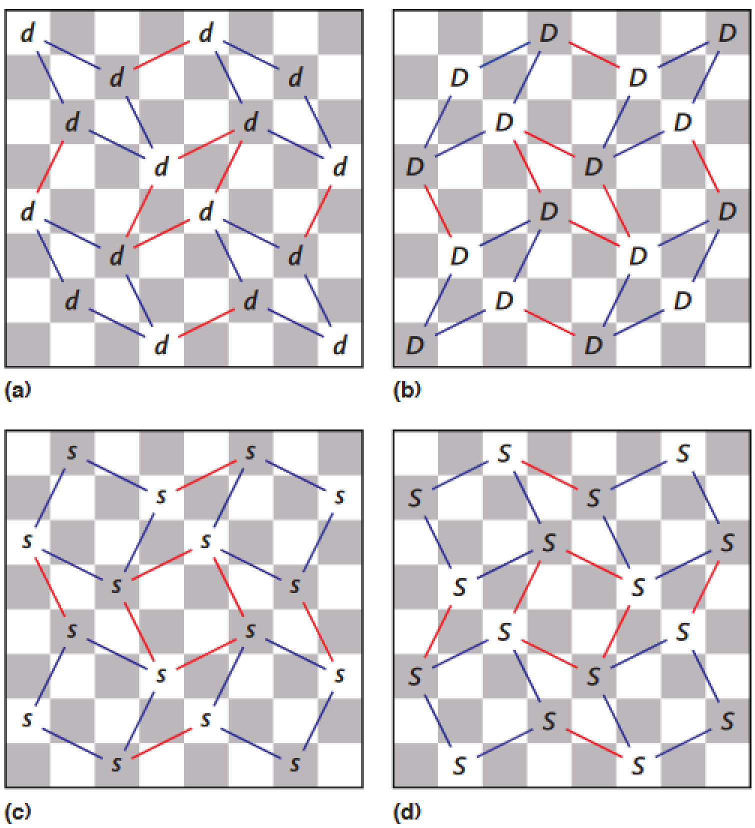
We will shift our attention to the Challenging Knight's Tour, and pose the question: for what combination of squares can we perform a knight's tour? First, note that on a chessboard, when a knight moves, the square it occupies changes color. Since a knight's tour takes 63 moves, a parity argument informs us that the start and end squares must be on opposite colors. We then ask: is the Challenging Knight's Tour always possible for any two chosen squares of opposite color? (Poll for guesses) The solution had been answered via brute-force algorithms in the past, however, in 2016, Arthur Benjamin and I demonstrated with a constructive proof that the tour is indeed always possible.

1.1 Reframing the problem

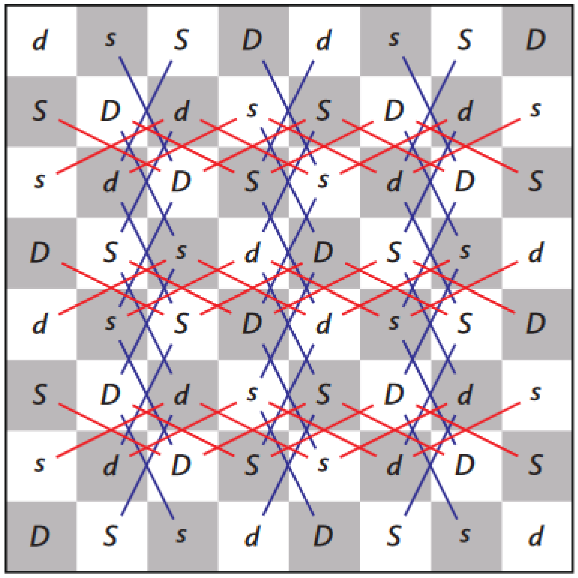
To solve this, we may construct a chessboard graph G with 64 vertices, each corresponding to a square on the chessboard, and with an edge existing between two vertices if and only if the squares are adjacent via knight-move. Additionally, we assign a color to each vertex, corresponding to the square's color on the board. Therefore adjacent vertices must be of opposite color. The problem of solving the challenging knight's tour is then equivalent to proving the following statement:

Theorem: (Benjamin, Miller) For any two opposite-colored vertices v_1, v_2 on G, there exists a Hamiltonian path which starts at v_1 and ends at v_2.

We will not cover the full proof here, but cover its main strategy. A key observation one may make is that G is decomposable into four isomorphic subgraphs, which are isomorphic to $G_{4,4}$, a 4×4 grid of vertices. We shall call these induced subgraphs the 4 *systems*. On the chessboard, these subgraphs' appearances allow us to further classify them into "diamond" and "square" systems. We denote these systems d, D, s, S.



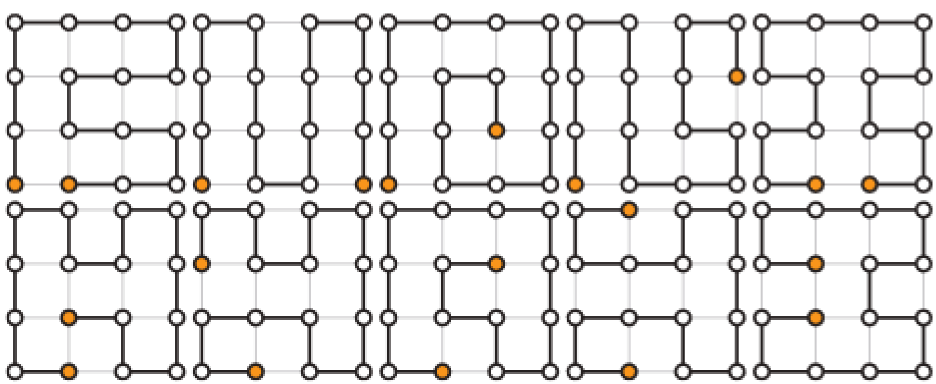
Moreover, if we consider all edges of G not belonging to one of the systems, we see that they form two shoelace patterns between systems of opposite type.



In particular, any vertex belonging to the 4×4 square in the middle of the chessboard is adjacent to vertices belonging to both systems of the opposite type. This suggests a strategy: complete the knight's tour by traversing a system, choosing wisely to end on a square which grants us access to both systems of opposite type, moving to the next system, and repeating until we are done. However, it is not immediately clear that in any situation, we can perform a traversal of $G_{4,4}$ where we can choose the end square.

Lemma: $G_{4,4}$ is traversable, that is, for any pair of distinguished vertices of opposite color in $G_{4,4}$, a Hamiltonian path exists which begins at one vertex and ends at the other.

Proof:

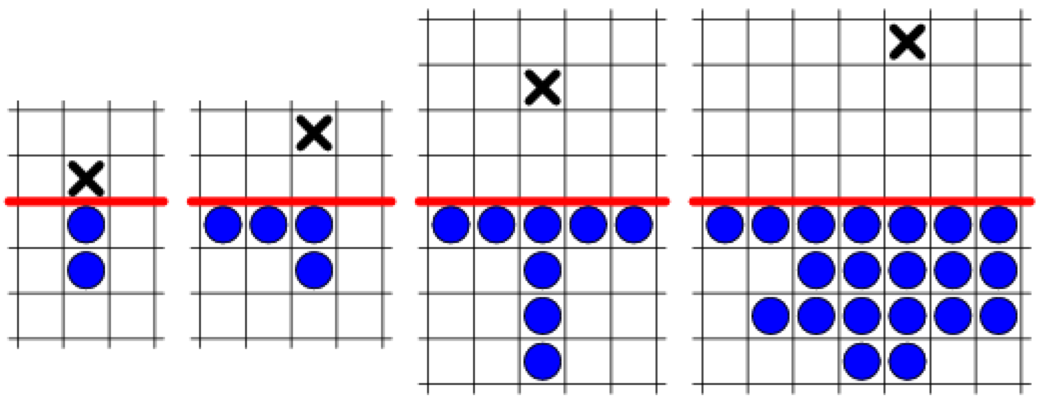


□

Therefore, we may always traverse a system, choosing wisely to end in the center of the board, so we may have access to both adjacent systems. Using this strategy, we immediately have as a result that if the start and end squares given of the challenging knight's tour belong to systems of opposite types, we may perform the tour. Additional care must be taken in the cases where the start and end squares are in either the same system, or different systems of the same type, and this part of the proof is omitted for brevity.

2 Conway's Soldiers

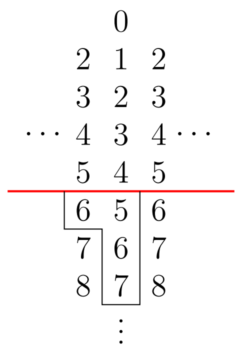
We now shift our attention to a different puzzle, another which contains jumps, but of a slightly different nature. This puzzle is known as Conway's Soldiers, and was devised by John Conway around 1961. We play on an endless grid/chessboard, which contains a horizontal line separating the board into two areas, the friendly territory and the enemy territory. The objective is to place pieces in your friendly territory, then make a sequence of moves that advances the pieces as far into enemy territory as possible. The pieces move similar to they do in peg solitaire - pieces move vertically or horizontally, are only allowed to hop over each other, and any pieces that is hopped over is removed from play.



To get 1 row into enemy territory, it's trivial to see we require 2 checkers minimum. (Let the audience play around and guess the next few). With some work, we may show that the minimum required for 2 rows is 4, for 3 rows is 8, and for 4 rows is 20. Now, would anyone like to guess how many checkers are required to make it 5 rows in? (You may have heard of the term "Combinatorial Explosion" used to describe Ramsey Theorey-esque problems before, perhaps this is an instance of that!)

Theorem: (Conway) It is **impossible** to reach level 5 with a finite amount of checkers!

Proof: Surprisingly, a rather elementary, and quite beautiful proof exists which proves this. Suppose to the contrary that there exists a square p on level 5 we are able to reach. Label that square 0, then for every square on row 4 or below, label that square by the number of steps it would take to move to p moving vertically or horizontally. Now, for the starting configuration which reaches square p, give each checker weight g^k, where k is the label of the square, and $g = (\sqrt{5} - 1)/2$. We say the *weight of the configuration* is the sum of the weights of all the checkers.



As an example, under this labelling, the row 2 configuration would have weight $(g^7 + g^6) + (g^6 + g^5) = g^5 + g^4 = g^3$.

Observe that $g^2 + g = 1$, or more generally, $g^n + g^{n-1} = g^{n-2}$. From this property, we observe that for any jump performed, the weight of the resulting configuration never increases.

- If a checker jumps to a square with lesser labelling, we replace two checkers of weights g^k, g^{k-1} with one checker of weight g^{k-2}, and since $g^k + g^{k-1} = g^{k-2}$, the weight of the new configuration is the same.
- If the checker jumps to a square with greater labelling, we replace two checkers of weight g^k, g^{k+1} with a checker of weight g^{k+2}, and since $g^k + g^{k+1} > g^{k+2}$ (as in general, $g^i > g^{i+1}$ for positive i), the weight of the resulting configuration increases.

Since the weight of the end configuration is at minimum $g^0 = 1$, our starting configuration must also have weight at least 1.

We compute the maximum possible weight of configuration, by bounding checker weights in each row. In row 1, we see that the maximum weight is bounded above by

$$(g^5 + g^6 + g^7 + \dots) + (g^6 + g^7 + \dots) = g^5/(1-g) + g^6/(1-g).$$

For $k \geq 2$, $g^k/(1-g) = g^{k-2}$ (this follows directly from the recursive property of g). Therefore,

$$g^5/(1-g) + g^6/(1-g) = g^3 + g^4 = g^2.$$

This bound is strict, as we only may place a finite amount of pieces. Similarly, in the second row, we find the weight is bounded by g^3, and in general, the n-th row is bounded by g^{n+1}. Therefore, the entire friendly area is strictly bounded by

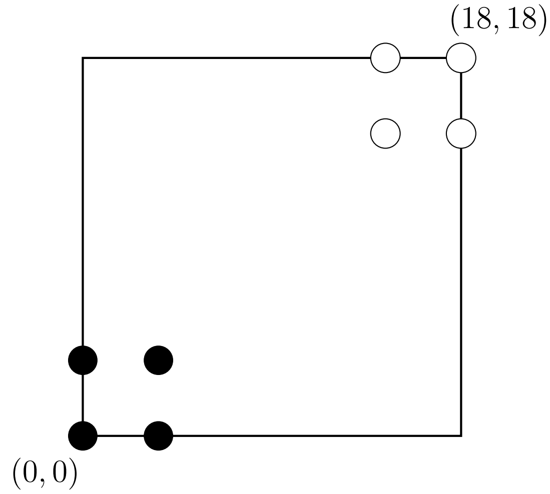
$$g^2 + g^3 + \dots = g^2/(1-g) = g^0 = 1.$$

This is a contradiction, since we required a weight of exactly 1 to reach the 5th row in enemy territory! Thus, no configuration exists. □

Surprisingly, however, Simon Tatham and Gareth Taylor have shown that it is possible to reach the 5th row if we are allowed to place an *infinite* number of checkers. Additionally, it has been shown that in the n-dimensional variant of the game, it is always possible to reach the $(3n-2)$th row, and the same argument we used may be applied to show that the $(3n-1)$th row is not reachable with a finite number of checkers.

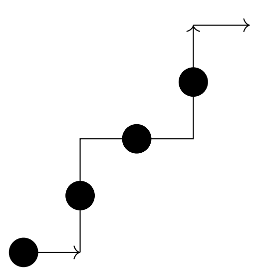
3 Optimal Leapfrogging

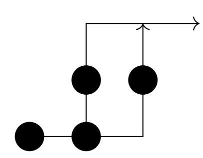
What if we allow the checkers to shift as well as hopping (but without the checker removal), like in Chinese Checkers (but on a Go board)? Then clearly the previous problem is trivial - so lets ask a different question, what is the most efficient way to move a bunch of checkers? Let's ask a more concrete problem:



The goal is to move 4 checkers from the bottom left corner to the top right corner in as few moves as possible. How should we go about this? (Pause for audience participation) If we play around with the problem a bit, we see that if we first shift the checkers into a well-chosen configuration, we can use the checkers' synergy to perform lots of hops. And as some more advanced combinatorial optimization results which I will not go into, imply, this is indeed the most efficient method of movement. Let's see if you can figure out any configurations! (Pause to let audience think)

One idea one may find is to first line the checkers up diagonally, then the backmost checker may hop over all of the others to reach the front of the serpent, then move into place. This method allows for a checker to jump, rather than shift, once every 3 moves. We compute that this method takes $4 + 3 \cdot 15 + 4 = 53$ moves. We shall call this strategy the *snake*





However, another strategy is to first align the checkers into a squiggle shape, like in Tetris. We notice this strategy, which we call the *serpent*, is advantageous, as we can repeatedly perform hops without needing to shift in between, and indeed, this is a more efficient method, which takes $1 + 16 \cdot 2 + 2 = 35$ moves! As it turns out, this is the fastest possible 4 piece configuration!

3.1 How fast can we go?

Why is the serpent faster? We notice that the serpent takes 2 moves to translate $(1, 1)$, while the snake takes 3 moves to translate $(1, 1)$. We can say that the snake has speed $2/3$, while the serpent has speed $2/2 = 1$. More generally, we define the **speed** of a configuration X to be $(a_1 + a_2)/m$, where X can translate itself in the direction (a_1, a_2) in m moves. Do you think there is a maximum speed, and if so, what could it be?

Lemma: The absolute upper bound of any configuration's speed is 1.

Proof: Let us suppose configuration X has p pieces. Suppose we can translate X to $X + (a_1, a_2)$ in m moves. If $a_1 + a_2 \leq 0$, the speed is ≤ 0, so without loss of generality, assume $a_1 + a_2 \geq 1$. We wish to show $m \geq a_1 + a_2$.

If $X = \{(x_1, y_1), \ldots, (x_p, y + p)\}$, define the *back border* of X as $\min(x_i + y_i)$ and the *front border* of X as $\max(x_i + y_i)$. Now observe that the back border and front border may only increase by 1 on each move. Moreover, we may only advance both the front and back border simultaneously if we perform a hop with a piece moving from the back border to ahead of the front border.

Since we move from X to $X + (a_1, a_2)$, the borders both increase by $(a_1 + a_2)$, so we require at least $(a_1 + a_2)$ moves. Thus $m \geq a_1 + a_2$, as desired. We note that $m = a_1 + a_2$ if and only if every move in the translation advances both the front border and back border. □

We say that a configuration which moves with speed 1 is *fast*. Some fast configurations include the serpent, the lone piece, and two adjacent pieces. Are there any others? (pause for wonder)

Theorem (Auslander, Benjamin, Wilkerson): The three aforementioned configurations are the *only* fast configurations - for any n-dimensional variant of this game.

Proof sketch: As noted before in the previous proof, a fast configuration must have the property that every move advances both the front and back border. Therefore, every border in the configuration must have exactly 1 piece. Assuming that $p > 2$, we analyze the piece placement of the translation in front of the starting front border, and by process of elimination show that the only possible configuration which can continuously perform jumps is the serpent.

The proof of this statement was originally shown by Arthur Benjamin, Joel Auslander, and Daniel Wilkerson in the 80s, as a concrete example of a more general phenomena which occurs in operations research. They additionally conjectured that for configurations with $p > 4$ pieces, the maximum attainable speed is $2/3$, which would make the p-snake a maximally efficient configuration. Through 2017 and 2018 I worked on this conjecture while working in industry.

Theorem (Miller): For $p = 3$ and $p > 4$, the maximum attainable speed of any configuration is 2/3 - for any n-dimensional variant of this game.

The proof will not be presented here, as it is quite lengthy and utilizes multiple new techniques not present in the fast configuration proof. The paper is still in progress, and has unfortunately entered limbo with COVID (it will hopefully be a joint publication with Benjamin). I'm hopeful that there exists a cleaner proof than the one that I've written.

4 A Combinatorial Characterization of Determinants

That's right, the determinant of a matrix counts something! To explain what it counts and prove why it counts what it does, I first need to introduce a method of counting which we call D.I.E., or Description-Involution-Exception.

4.1 D.I.E.

To illustrate the method of D.I.E., we will first prove a simple identity. For $0 \leq m \leq n$, we wish to find a closed form of

$$\sum_{k=0}^{m} \binom{n}{k} (-1)^k.$$

As an aside, there does not exist a "nice" closed form of the non-alternating sum $\sum_{k=0}^{m} \binom{n}{k}$. Combinatorially, this sum counts the number of subsets of $\mathcal{N} = \{1, \ldots, n\}$ of even cardinality, minus the number of subsets of $\mathcal{N}$ of odd cardinality. Observe that there exists a bijection between subsets of even cardinality and of odd cardinality, given by "toggling the 1":

$$\pi : M \subseteq \mathcal{N} \mapsto \begin{cases} M \oplus \{1\} & \text{if } 1 \notin M \\ M \setminus \{1\} & \text{if } 1 \in M \end{cases}$$

As another aside, this bijection provides one of many proofs that $\sum_{k=0}^{n} \binom{n}{2k} = 2^{n-1}$. Perhaps in a further talk, I will discuss sums of the form $\sum \binom{n}{a+rk}$ for $r \in \mathbb{N}, 0 \leq a \leq k$.

Therefore, for every mutual even-odd subset pair (pairs of subsets which are identical, except for one subset additionally containing 1), their terms in the sum cancel. Thus the only subsets of $\mathcal{N}$ which are not cancelled are those for which "toggling the one" results in a subset not considered by the sum. These subsets are precisely those of cardinality m and which do not contain 1. There are $\binom{n-1}{m}$ of these, all with parity $(-1)^m$. Thus,

$$\sum_{k=0}^{m} \binom{n}{k} (-1)^k = (-1)^m \binom{n-1}{m}. \quad \square$$

Corollary: $\sum_{k=0}^{n} \binom{n}{k} (-1)^k = 0$

In this proof, we utilized the technique of D.I.E.

- **Description**: Given an alternating sum $\sum(-1)^k a_k$, we consider what the non-alternating sum $\sum a_k$ counts.
- **Involution**: We find an *involution*, a bijection between most even and odd k-terms.
- **Exception**: We count the number of exceptions to the involution.

4.2 The Problem of the DeterminedAnts

Recall that given a $n \times n$ matrix $A = a_{i,j}$, the *determinant* of A is defined as

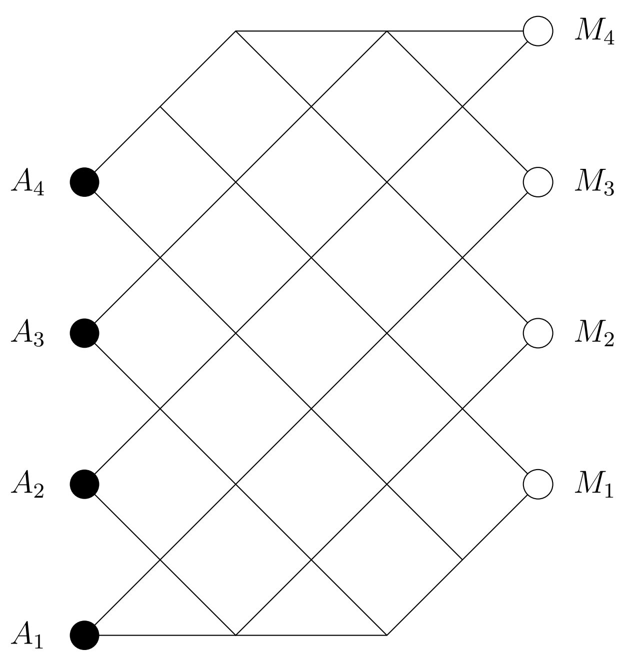
$$\det A = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}.$$

This is an alternating sum, perfect for applying D.I.E.! This means we also need to consider the non-alternating version of the sum.

Definition: Define the *permanent* of A to be

$$\operatorname{per} A = \sum_{\sigma \in S_n} (\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}$$

We now introduce **The Problem of the DeterminedAnts**! Consider a directed graph $\mathcal{D}$, with n designated starting nodes A_i, the ants, and n ending nodes M_j, the morsels. The objective of the "puzzle" is to lead each ant to a morsel so that no two ants end at the same morsel. We require of the digraph that for $i \neq j$, any paths from A_i to M_j intersect paths from A_j to M_i, and that the graph is acyclic.
As an example, we give the following digraph, where ants are only allowed to move to the right:



Definition: The *Ant Matrix* of $\mathcal{D}$, A, stores in $a_{i,j}$ the number of ways ant A_i can reach morsel M_j, i.e. the number of paths from A_i to M_j.

For example, the ant matrix of the above graph $\mathcal{D}$ is:

$$A = \begin{pmatrix} 14 & 6 & 1 & 0 \\ 20 & 15 & 6 & 1 \\ 15 & 20 & 15 & 6 \\ 6 & 15 & 20 & 14 \end{pmatrix}$$

Now, how many ways are there to solve the puzzle? (pause to see if anyone sees it)

Theorem: The *permanent* of A counts the number of pairs of simultaneous paths from a unique ant to a unique morsel.

Proof: Suppose $\mathcal{D}$ has n ants and morsels. Then, each $\sigma \in S_n$ is in 1 to 1 correspondence to a unique pairing of ants to morsels:

$$\pi : \sigma \mapsto \{(1, \sigma(1)), (2, \sigma(2)), \ldots, (n, \sigma(n))\}.$$

Given a pairing of ants to morsels $\pi(\sigma)$, the number of pairings of paths of the form $(1, \sigma(1)), \ldots, (n, \sigma(n))$ is given by:

$$a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}.$$

Therefore, the number of ways to guide the ants is given by:

$$\sum_{\sigma \in S_n} a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} = \operatorname{per} A. \quad \square$$

Now suppose we change the rules of the puzzle to additionally require that no two ants' paths touch. Now, how many ways there are to solve the puzzle? (pause again)

Theorem: (Gessel, Viennot) The *determinant* of A counts the number of simultaneous paths from a unique ant to a unique morsel, for which no two paths touch.

Proof: This result follows from D.I.E. We have a characterisation of the non-alternating sum per A, which counts all pairings of paths, so det A counts the pairings of paths corresponding to even permutations, minus the pairings of paths corresponding to odd permutations. We wish to find an involution between even and odd pairings.

For a valid pairing of paths, consider the smallest path (ordered by ant number) which intersects another path, and take its first touching. Create a new valid pairing of paths by tail swapping the two paths, that is, if $A_i \to M_j$ and $A_k \to M_l$, the new pairing has $A_i \to M_l$ and $A_k \to M_j$. This new path has a corresponding permutation with opposite sign! Moreover, note that performing the process again yields the starting path, since the tail swap does not change the first point of intersection of the path.

Therefore, we have found a sign changing involution! Moreover, it is easy to classify the exceptions to the involution - it is precisely the pairings of paths which have no intersections. These all correspond to $1 \in S_n$, an even permutation, by how we restricted the graph. Thus, we conclude $\det A$ counts the number of valid pairings of paths from ants to morsels which do not touch. $\square$

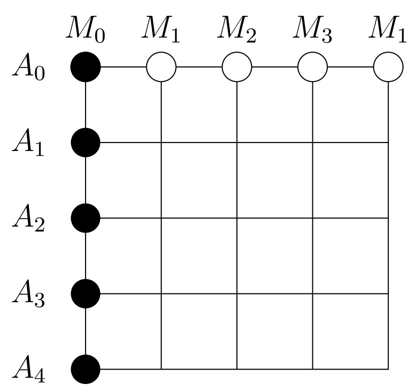
This is a gameified corollary of the Lindström–Gessel–Viennot lemma, proven in 1985 by Gessel and Viennot, which is a more general statement concerning the situation when there exist nonintersecting pairings of paths which do not correspond to $1 \in S_n$. In this case, the determinant counts the even nonintersecting pairings minus the odd nonintersecting pairings. This lemma shows up in the proof of equivalence of two definitions of Schur polynomials, symmetric polynomials which show up in the study of Young tableaux.

As a corollary, we may determine the determinant of Pascal's matrix P_n - generated in the same way as Pascal's triangle. More precisely, $p_{i,j} = \binom{i+j}{j}$ (with the indexing starting at 0). For example,

$$P_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{pmatrix}.$$

Theorem: For $n \geq 1$, $\det P_n = 1$.

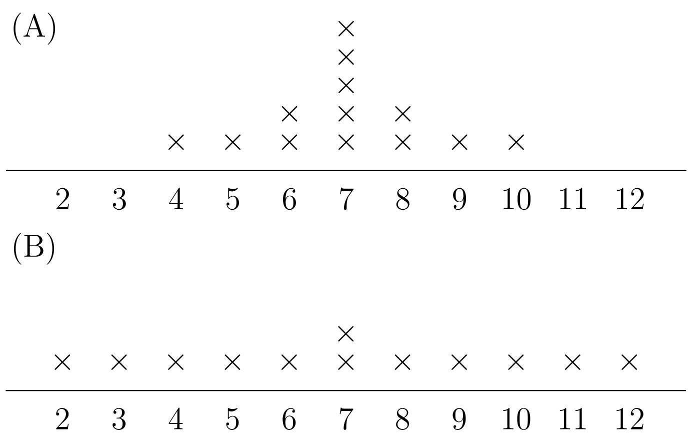
Proof: Consider the grid graph $G_{n,n}$ (pictured here is $G_{4,4}$)



Let us additionally suppose $G_{n,n}$ is a digraph, with directions only moving up and right, and set up the ants and morsels for the problem of the DeterminedAnts like in the diagram. Observe that there are precisely $\binom{i+j}{j}$ paths for ant A_i to reach morsel M_j, as the ant must make $i + j$ moves, j of which must move to the right. Therefore, P_n is the ant matrix. However, it is also clear that there is exactly 1 nonintersecting pairing of paths, so we conclude $\det P_n = 1$. $\square$

5 Knock Em Down

We shift our attention to a 2 player game, and one that doesn't involving moving pieces, at that. This time, our study begins to encroach game theory territory, and we switch our focus to probabilistic, not deterministic, questions. This suggests things may get wonky...
To play Knock Em Down, an old betting game (apparently?), we have two 6-sided die, and each player is given 12 tokens to place on a board, consisting of columns labeled from 2 to 12. When a number is rolled, players remove a token above the respective column, if there is still a token there. The first player to remove all tokens is the victor! For an example game between players A and B:



Who do you suppose wins? (pause and let audience vote/think) As it turns out, A beats B 75% of the time, B beats A 16% of the time, and there is a draw 9% of the time.

5.1 Simpler Cases

Our goal is to determine optimal play in Knock Em Down, but probabilistically, this game is rather cumbersome to study, so lets study a simpler version, this time with 1 die, 5 tokens, and fewer columns, with distribution as demonstrated below:

	(A)	(B)
1		
2, 3	×	
4, 5, 6	×× ××	×× ×××

In this situation, we see that player (A) should be favored - if a 2 or 3 is rolled before five 4s, 5s, or 6s have been rolled, then (A) automatically wins. Otherwise, (B) wins. It is impossible to reach a draw! It is not so hard to compute here that the odds that (A) wins are roughly 92%. However, if we distribute the checkers somewhat differently, like so:

	(A)	(B)
1	×	×
2, 3	××	×
4, 5, 6	××	×××

Who do we think wins here? This situation is a bit more complex. We notice that if a 2/3 is rolled twice, then (A) has an advantage, since one of those rolls benefits (A) but not (B). On the other hand, if a 4/5/6 is rolled thrice, then (B) has an advantage for similar reasoning. We can also have a tie if a 1 is not rolled until both the 2/3 and 4/5/6 boxes are cleared out. As it turns out, these placements are very evenly matched, with (A) winning 35.3% of the time, and (B) winning 33.6% of the time. A draw happens nearly a third of the time!

Note that in both setups, (A)'s placement looked a bit closer to the histogram of probabilities than (B). In the first game, (A)'s setup was considerably closer to the histogram, but in the second game, the setups were both off by about the same amount. Perhaps this indicates that the optimal strategy should be to follow the histogram of probabilities as close as possible! Do you think this is the case? (pause for thought)

Let's test this hypothesis by setting up a game where it is actually possible to follow the histogram as closely as possible. Suppose we have 10 tokens and 4 probability outcomes, of distribution $P(.1, .2, .3, .4)$. Consider the following placements:

	(A)	(B)
1	$\times$	
2, 3	$\times\times$	$\times\times$
4, 5, 6	$\times\times\times$	$\times\times\times$
7, 8, 9, 10	$\times\times$ $\times\times$	$\times\times$ $\times\times\times$

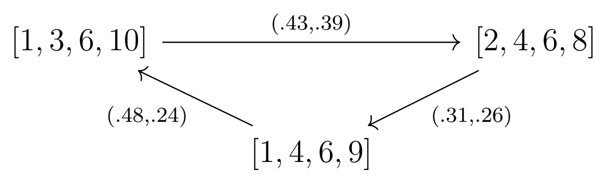
(A) follows the histogram of probabilities exactly. However, the expected value of the number of rolls for (B) to be cleared, $E[X_B] = 16.3$, while $E[X_A] = 17.7$. In fact, it may be shown that $E[X_B]$ is **minimal** over all allocations. Therefore our hypothesis fails! (pause)

Or does it? We also observe that if a 1 is rolled before the game is over, (A) cannot lose! On the other hand, the only win condition (B) has is having a 7/8/9/10 rolled 5 times before a 1 is rolled. Despite (B) on average taking less time to finish, (A) beats (B) more often, 36% to 23%, with a draw the most likely outcome, at 41%. In fact, it may be shown that (A) has an advantage over all possible allocations. We say that "(A) is an **emperor**." What a counter-intuitive phenomenon!

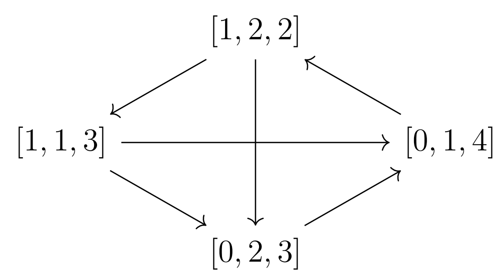
We've got some questions to ask: the first being, does this phenomena occur on even smaller games? Yes! We may have a game with probability outcomes $P(1/3, 2/3)$ and 2 tokens. Then, the distribution $A = [1, 1]$ has $E[X_A] = 3.5$, while $B = [0, 2]$ has $E[X_B] = 3$ (the computations of these may be performed by anyone who has taken an introduction to probability course, most likely). However, B only wins when the 2/3 is picked twice in a row, which has probability 4/9 - so it loses to A a majority of the time! Thus A is the emperor. (Note we have excluded $[2, 0]$, but it is not hard to show that this configuration is suboptimal in all ways)

5.2 Non-transitivity

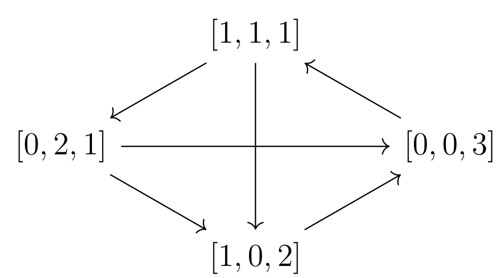
We next ask - for any variant of the game, is there always an emperor? To answer this, let us consider the game with 20 tokens given by the distribution $P(.1, .2, .3, .4)$ as before. With some work, we may show that $A = [1, 3, 6, 10]$ is minimal, and $B = [2, 4, 6, 8]$ is relatively slow. We also find A beats B 43% of the time, and B beats A 39% of the time. However, we may also consider $C = [1, 4, 6, 9]$. With some more work, we find the following situation occurs:



There is no emperor in this setup! (taking for granted that these are the three most optimal placements) We say this is a **non-transitive** game. Going back to the earlier game with 5 tokens and $P(1/6, 2/6, 3/6)$, it may be shown that this setup is also nontransitive, with the following diagram:



In fact, a non-transitive game with even less pieces exists! Now we restrict to 3 tokens. Under these rules, $[0, 1, 2]$ is both the emperor and minimal. However, if we ban $[0, 1, 2]$, then we find:



It is unknown at this time if there exist other nontransitive 3 token games. However, we do know:

Theorem: With 2 tokens, there is no nontransitive situation.

Theorem (Nelson): As the number of tokens grows, the probability of an emperor existing goes to 0.

An open problem is: given a game with t tokens and probability vector $P(p_1, \ldots, p_k)$, find a minimal allocation x^* and emperor $\hat{x}$ if one exists. Though in general this is a very difficult problem, there are some cases which are easier to work out! The first example is coin games - a game with a probability vector containing 2 outcomes. As an example, for $P(2/3, 1/3)$ on $t = 9$ pieces, $\hat{x} = (6, 3)$ and $x^* = (7, 2)$.

Theorem: The minimal allocation of the game with probability distribution $P(p, 1-p)$ and t checkers is $x^* = (m, t-m)$, where m is the pth percentile of $\text{Bin}(t, p)$.

5.3 Choosing Wisely

We now know that for games with large numbers of pieces t, emperors are rare or nonexistent. In fact, given a reasonable distribution (A), we may find some distribution (B) which defeats (A) by **undercutting** (A). We demonstrate via example:

Let $P(1/3, 1/3, 1/3)$ with $t = 3000$. Set (A)$=[1000, 1000, 1000]$. We undercut (A) by creating (B)$=[999, 999, 1002]$. Why does this work? Consider this - let's play until (B) is exhausted. We ask - what are the odds that bin 3 was emptied last? It may be computed to be around .36. However, if Bin 1 or Bin 2 finished last, then (A) cannot be done, so we see that indeed, (B) is heavily favored. However, by a similar argument we see that (C)$=[998, 1001, 1001]$ beats (B), and (A) beats (C)! So we have used undercutting to create a cycle. In general, this may be used to deny existence of an emperor.

On the other hand, for any game, a minimal allocation *must* exist, though it may not be unique! Fortunately, narrowing down the list of potentially minimal allocations is possible.

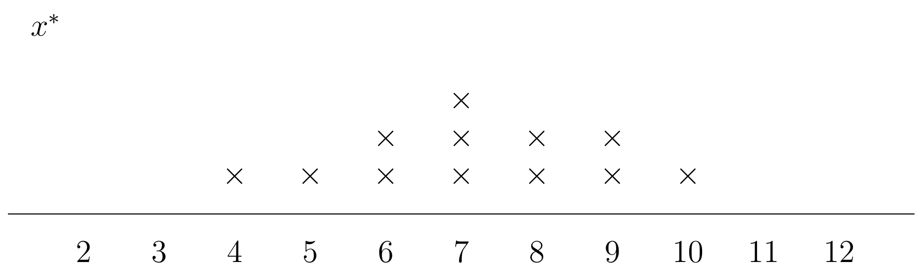
Theorem: Let $x^* = (x_1, \ldots, x_n)$ be a minimal allocation for a t token game with $P(p_1, \ldots, p_n)$ probability vector. Then x^* satisfies:

(a) If $p_i < p_j$, $x_i \leq x_j$.

(b) If $p_i = p_j$, $|x_i - x_j| \leq 1$.

(c) If $p_i < p_j$, then $(x_i - 1)/x_j < p_i/p_j$.

Applying this to our original setup (2 dice, 12 tokens) reduces the number of potential minimal allocations from 646,646 to 49! As it turns out, a minimal allocation is:



Some final conjectures which at this time are (probably still) open:

- If x^* is minimal for a t token game, then there is a minimal allocation for the $(t + 1)$ token game containing x^*.
- In the original game, x^* seems to be an emperor as well. It is a *local* emperor in that it beats all allocations obtained by shifting one token. Note that a local emperor in general is not a global emperor. On the other hand, one open question is: when is a local minimal allocation is a global minimal allocation?
- The minimal allocation cannot differ too much from P. Precisely,

$$\lim_{t \to \infty} \frac{x^*(t)}{t} = P.$$