A "BRIEF" SUMMARY OF CURRENT RESEARCH INTERESTS

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If anything here seems interesting, or you have any questions/comments, I'm happy to chat!

Permutation modules, endotrivial complexes, and twisted cohomology. Let G be a finite p-group, and k a field of prime characteristic p, and let $\mathcal{K}(G) := \mathrm{K}_b(p\text{-perm}(kG))$ denote the homotopy category of p-permutation modules, a tensor-triangulated category. Balmer–Gallauer deduced the Balmer spectrum $\mathrm{Spc}(\mathcal{K}(G))$ of $\mathcal{K}(G)$, the universal support of the category [BG25]. In [Mil25c], I give a more general construction generalizing Balmer–Gallauer's *permutation twisted cohomology ring* $\mathrm{H}^{\bullet\bullet}(G)$ constructed in [BG25, Part II], using the classification of endotrivial complexes, the invertible objects of $\mathcal{K}(G)$, which I previously deduced in [Mil25a]. The ring comes with a canonical comparison map $\mathrm{comp}_G\colon \mathrm{Spc}(\mathcal{K}(G)) \to \mathrm{Spec}^h(\mathrm{H}^{\bullet\bullet}(G))$ which I show is always injective, and an open immersion when $\mathrm{H}^{\bullet\bullet}(G)$ is Noetherian. I also constructed an open cover of $\mathrm{Spc}(\mathcal{K}(G))$ indexed by conjugacy classes of subgroups of G; under this open cover, if $\mathrm{H}^{\bullet\bullet}(G)$ is Noetherian then $\mathrm{Spc}(\mathcal{K}(G))$ has (Dirac) scheme structure. There are a lot of moving parts here, so let me hone in on a few important points and remaining questions.

There are a few key features and constructions which make all this possible. First is restricting to what I've termed *effective endotrivials*, which are endotrivial complexes which arise from the Bredon homology of a representation sphere, up to shift. These complexes have additional nice structure: they arise from chain complexes of free modules over the orbit category Γ_G, which allows us to say more about their differentials and maps $k \to C$ from the tensor unit into them, which Gallauer terms *global sections* [Gal25]. A special kind of global section is what I've called a *forerunner*: a map which, after applying the modular fixed point functor $\Psi^H\colon \mathcal{K}(G) \to \mathcal{K}(N_G(H)/H)$ (perhaps more familiarly, the Brauer quotient), becomes a quasi-isomorphism. These are constructed explicitly, in such a way that they do not become quasi-isomorphisms for other subgroups when possible. However the construction is quite ad-hoc, not natural or universal in any given sense. These maps do not seem to arise topologically (namely they are *coaugmentation* maps, when the differentials of an effective endotrivial are *augmentation* maps), but I expect that there is a topological interpretation of such maps that would allow us to better understand these things and prove certain properties one would expect. Additionally, it would further clarify the open cover we construct, since it is formed by essentially inverting these maps in the right way.

Another obvious lingering question is Noetherianity of twisted cohomology, since that's the last piece for deducing that $\mathrm{Spc}(\mathcal{K}(G))$ is a Dirac (2?-)scheme. Informally, the twisted cohomology ring is simply the ring consisting of all global sections into effective endotrivials. I believe that it is in general a hard question, and that what's missing is a fact about the differentials of effective endotrivials. This may again be where a topological point of view will be necessary, since the behavior of these endotrivials past small enough groups is still fairly nebulous.

There are more projects, which I'll briefly mention:

- Now that we have an open cover, I'd like to try to reinterpret the Borel-Smith functions (the functions which give us the classification of endotrivials, and which govern dimension functions of representation spheres) in terms of descent data on $\mathcal{K}(G)$. Gallauer suggested a descent spectral sequence should work, and now that I'm finally done with the preprint in question, I can start diving into that.

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- In joint work with Gomez, we're working on extending the classification of endotrivials to integral base rings more generally, over arbitrary Noetherian base rings. Then the question is - can twisted cohomology be extended to produce the same results? That would be a significant leap - one may want to simply restrict to the complexes arising from Bredon homology over a base.

Noncommutative tensor-triangular geometry. I've been active in the development of noncommutative tensor-triangular geometry, initiated by Nakano–Vashaw–Yakimov [NVY22] and Buan–Krause–Solberg [BKS07]. In particular, I've been interested in the question of functoriality of the noncommutative Balmer spectrum (or lack thereof) and the tensor product property [Mil25b], and the classification of various types of thick tensor ideals [DM25, Mil25d] using frame theory and Stone duality in the vein of [KP17, GS23].

[Mil25b] is a hodgepodge of results, some about the *complete prime spectrum* $\mathrm{Spc}^{cp}(\mathcal{K})$ of a monoidal triangulated category $\mathcal{K}$ (i.e., an exact but not-necessarily symmetric tensor product). A complete prime is what you get if you write down the symmetric version of a prime thick $\otimes$-ideal, and it turns out these are precisely the radical primes, and classify all such radical thick $\otimes$-ideals. I also consider a few questions about functoriality of the usual noncommutative Balmer spectrum, classifying one-sided primes (it's generally hopeless), and deduced a "universal functorial support theory" following the strategy of [Rey12]. Finally, I deduced when an induced map on the complete prime spectrum is surjective. This requires an additional property for the target monoidal triangulated category $\mathcal{K}$, *duoness*, which Nakano–Vashaw–Yakimov showed implies $\mathrm{Spc}(\mathcal{K})$ has the tensor product property. I showed in a short paper [Mil25c] that the converse to this theorem fails by showing the existence of a semisimple replete subcategory of Khovanov's original construction of the Heisenberg category. This has spurned my interest in finding more examples, commutative and noncommutative, of intersections between diagrammatic categories (e.g. categories whose morphisms are defined by strand diagrams) and tensor-triangular geometry.

In [DM25], De Deyn and I proved the noncommutative analogue of Balmer's classification theorem [Bal05] utilizing Stone duality. We let $\mathrm{Spc}(\mathcal{K})^\nu$ denote the *pseudo-Hochster dual* of the Balmer spectrum: the supports form an open base of the topology. Then one has an order-preserving bijection (a frame isomorphism) between the opens of $\mathrm{Spc}(\mathcal{K})^\nu$ and the semiprime thick tensor ideals of $\mathcal{K}$. Such a result in full generality was known in specific cases (e.g. $\mathrm{Spc}(\mathcal{K})$ Noetherian [Row24], $\mathrm{Spc}(\mathcal{K}) = \mathrm{Spc}^{cp}(\mathcal{K})$ [MR23, Mil25b], $\mathcal{K}$ has a thick generator [NVY23]) but [HV25] gave an example where thick tensor-ideals were parametrized by a collection of subsets of $\mathrm{Spc}(\mathcal{K})$ strictly larger than the Thomason subsets; our result reconciles these results. We moreover examine the conditions of what needs to happen for the classification to behave as in the tensor-triangular case, and show that these hold in a few additional settings. We are currently considering follow-up questions, including the relation between tensor-triangular geometry and tensor-*exact* geometry as hinted by Krause [Kra24], where other spectra fit into the picture (e.g. Matsui), and better understanding when the conditions we determined arise.

Finally, Vashaw and I have initiated a project following up on his work with Sølberg–Witherspoon [SVW25], in which the authors consider the noncommutative tensor-triangular geometry of bimodules over unipotent Hopf algebras. Our goal is to determine the noncommutative tensor-triangular geometry of kP, for P a p-group and k a field of characteristic p. A long-term goal I'd like is to determine the noncommutative tensor-triangular geometry of p-permutation bimodules, but this probably requires a better understanding of what happens in the stable or derived module category setting first.

Miscellaneous. In joint work with Harman, we're broadly investigating the tensor-triangular geometry of certain oligomorphic groups and interpolation categories, building on his work with Snowden [HS25] and a recent preprint by Coulombier–Snowden [CS26]. Computations of *tensor* *ideals* and *thick ideals*, different notions than that of a *thick tensor ideal*, have been computed previously; our hope is that we can reconcile differences between the three.

I've had a broad interest in understanding other tensor-triangular geometric phenomena, and am still learning more there. One concept that's piqued my curiosity is Krause's development of *central subcategories* [Kra23]. I'm also interested in the applications of frame theory towards tensor-triangular geometry in the rigidly-compactly-generated setting.

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