Notes on tensor induction of chain complexes

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1 Tensor induction of modules

Definition 1.1. (a) Let H be a finite group and n a non-negative integer. Then S_n acts on the group H^n via $\pi \cdot (h_1, \dots, h_n) := (h_{\pi^{-1}(1)}, \dots, h_{\pi^{-1}(n)})$. The resulting semidirect product $H^n \rtimes S_n$ is called *the wreath product*, and denoted $H \wr n$.

(b) Let $H \leq G$ be finite groups with $[G : H] = n$, and fix a set of coset representatives $\{g_1, \dots, g_n\}$. Then there is an embedding $i_G^{G\wr n} : G \hookrightarrow H \wr n$ given by

$$g \mapsto \pi(h_1, \dots, h_n) = (h_{\pi^{-1}(1)}, \dots, h_{\pi^{-1}(n)}; \pi), \quad \text{where } gg_i = g_{\pi(i)}h_i.$$

The embedding is not canonical, as it relies on a choice of coset representatives. However, all such embeddings are conjugate subgroups.

(c) Given an RH-module M, denote by $M \wr n$ the $R[H \wr n]$-module $M \otimes_R \cdots \otimes_R M = M^{\otimes n}$ as R-module, with $H \wr n$-action given by

$$(h_1, \dots, h_n; \pi) \cdot m_1 \otimes \cdots \otimes m_n := h_1 \cdot m_{\pi^{-1}(1)} \otimes \cdots \otimes h_n \cdot m_{\pi^{-1}(n)}.$$

The restriction induced by the embedding $i_G^{H\wr n} : G \hookrightarrow H \wr n$ produces an RG-module, denoted $\operatorname{Ten}_H^G M$. Precisely, the action is

$$g \cdot (m_1 \otimes \cdots \otimes m_n) = h_{\pi_g^{-1}(1)} \cdot m_{\pi_g^{-1}(1)} \otimes \cdots \otimes h_{\pi_g^{-1}(n)} \cdot m_{\pi_g^{-1}(n)}.$$

This is the *tensor induced RG-module obtained from M*. It follows that this construction is independent up to isomorphism of coset representatives of G/H.

(d) Similarly, given a finite H-set X, we define a $H \wr n$-set $X \wr n$ via

$$(h_1, \dots, h_n; \pi) \cdot (x_1, \dots, x_n) := (h_1 \cdot x_{\pi^{-1}(1)}, \cdots, h_n \cdot x_{\pi^{-1}(n)}).$$

The restriction by the embedding $i_G^{H\wr n} : G \hookrightarrow H \wr n$ produces a G-set, denoted $\operatorname{Ten}_H^G(X)$, the *tensor induced G-set obtained from X*. It is clear from the construction that for any H-set X, $\operatorname{Ten}_H^G(R[X]) \cong R[\operatorname{Ten}_H^G X]$.

Remark 1.2. (a) Tensor induction of modules is multiplicative, in that it satisfies the following identity: for any RH-modules M, N,

$$\operatorname{Ten}_H^G(M) \otimes_R \operatorname{Ten}_H^G(N) \cong \operatorname{Ten}_H^G(M \otimes_R N).$$

More generally, $(M \wr n) \otimes_R (N \wr n) \cong (M \otimes_R N) \wr n$, which follows after not applying restriction. However, it is not additive, that is,

$$\operatorname{Ten}_H^G(M) \oplus \operatorname{Ten}_H^G(N) \not\cong \operatorname{Ten}_H^G(M \oplus N)$$

. Similarly, for any H-sets X, Y,

$$\operatorname{Ten}_H^G(X) \times \operatorname{Ten}_H^G(Y) \cong \operatorname{Ten}_H^G(X \times Y).$$

(b) Tensor induction (and more generally, $- \wr n$) of modules is functorial in the following way: for any $f : M \to N$, define $\operatorname{Ten}_H^G f : \operatorname{Ten}_H^G M \to \operatorname{Ten}_H^G N$ by:

$$\operatorname{Ten}_H^G f(m_1 \otimes \cdots \otimes m_n) := f(m_1) \otimes \cdots \otimes f(m_n).$$

(c) By Dress's theory of "algebraic maps," tensor induction on H-sets can be extended uniquely to a multiplicative map $B(H) \to B(G)$ which coincides with tensor induction on virtual H-sets with positive coefficients. The formula can be expressed as follows: given $[S] - [T] \in B(H)$,

$$\operatorname{Ten}_H^G([S]-[T]) = [\operatorname{Ten}_H^G S] - \big([\operatorname{Ten}_H^G(S \sqcup T)] - [\operatorname{Ten}_H^G S]\big) + \big([\operatorname{Ten}_H^G(S \sqcup T \sqcup T) - 2[\operatorname{Ten}_H^G(S \sqcup T)] + [\operatorname{Ten}_H^G S]\big) - \cdots$$

In Curtis & Reiner's "Methods of Representation Theory Volume 2" a simplified formula (80.49) is given:

Let $X = [S] - [T]$ for H-sets S, T, and for each $i \in \{0, \ldots, n\}$, let V_i be the G-subset of $\operatorname{Ten}_H^G(S \sqcup T)$ consisting of all elements having exactly i elements from S and $n - i$ entries from T. Then,

$$\operatorname{Ten}_H^G([S] - [T]) = [V_n] - [V_{n-1}] + \cdots + (-1)^n [V_0] \in B(G).$$

However, this is false in general, and in fact is not even a well-defined formula. For example, one may check that applying this formula to $0 = [1/1] - [1/1] \in B(1)$ yields:

$$0 = \operatorname{Ten}_1^{C_2} 0 = \operatorname{Ten}_1^{C_2}[1/1] - [1/1] = 2[C_2]/[C_2] - [C_2/1].$$

(d) Similarly, tensor induction on p-permutation RH-modules can be extended uniquely to a multiplicative map $T(RH) \to T(RG)$ which coincides with tensor induction on virtual p-permutation RH-modules with positive coefficients. The formula follows analogously. Later on, we will give a criteria for when the above incorrect formula holds.

We begin by proving a transitive property of tensor induction. Denote by $S_{(a,b)}$ the symmetric group acting on the set $\{(i,j) \in \mathbb{Z}^2 : 1 \leq i \leq a, 1 \leq j \leq b\}$. It is isomorphic to S_{ab}, but not canonically, by choosing an ordering of the set.

Proposition 1.3. (a) $S_n \wr m \hookrightarrow S_{(m,n)} \cong S_{mn}$ via the inclusion

$$\begin{aligned}(\sigma_1, \ldots, \sigma_m; \pi) &\mapsto G(\sigma_1, \ldots, \sigma_m; \pi) := \big((i,j) \mapsto (\pi(i), \sigma_{\pi(i)}(j))\big) \in S_{(m,n)} \\ &\mapsto G'(\sigma_1, \ldots, \sigma_m; \pi) := (n(i-1)+j) \mapsto n(\pi(i)-1) + \sigma_{\pi(i)}(j) \in S_{mn}\end{aligned}$$

This induces an inclusion $(K \wr n) \wr m \hookrightarrow K \wr nm$ given by:

$$\big((k_1^1, \ldots, k_n^1; \sigma_1), \ldots, (k_1^m, \ldots, k_n^m; \sigma_m); \pi\big) \mapsto (k_1^1, \ldots k_n^1, k_1^2, \ldots, k_n^m; G'(\sigma_1, \ldots, \sigma_m; \pi))$$

(b) Let $K \leq H \leq G$ be finite groups, with $[H : K] = n$ and $[G : H] = m$. Fix sets of coset representatives $[H/K] = \{h_1, \ldots, h_n\}$ and $[G/H] = \{g_1, \ldots, g_m\}$. Then with the chosen set of coset representatives $[G/K] = \{g_1h_1, g_1h_2, \ldots, g_1h_n, g_2h_1, \ldots g_mh_n\}$, the prior injective group homomorphism $i : (K \wr n) \wr m \hookrightarrow K \wr mn$ makes the following diagram commute:

$$\begin{array}{ccc} G & \xrightarrow{i_G^{K\wr mn}} & K \wr mn \\ {\scriptstyle i_G^{H\wr m}}\downarrow & & \uparrow{\scriptstyle i} \\ H \wr m & \xrightarrow[i_H^{K\wr n}\wr m]{} & (K \wr n) \wr m \end{array}$$

where $i_H^{K\wr n} \wr m$ is the map induced by $i_H^{K\wr n}$ on all copies of H in $H \wr m$, i.e. the image under the functor $(-) \wr m$.

Proof. The first statement is tedious but straightforward to verify. Let $g \in G$. Then under $i_G^{H\wr m}$,

$$g \mapsto \pi(h'_1, \dots, h'_m) = (h'_{\pi^{-1}(1)}, \dots, h'_{\pi^{-1}(m)}; \pi) \in H \wr m,$$

where $gg_i = g_{\pi(i)}h'_i$ defines π and each $h'_i \in H$. Then composed with $i_H^{K\wr n} \wr m$,

$$\begin{aligned} g &\mapsto \big(\sigma_{\pi^{-1}(1)} \cdot (k_1^{\pi^{-1}(1)}, \dots, k_n^{\pi^{-1}(1)}), \dots, \sigma_{\pi^{-1}(m)} \cdot (k_1^{\pi^{-1}(m)}, \dots, k_n^{\pi^{-1}(m)}); \pi\big) \\ &= \big((k^{\pi^{-1}(1)}_{\sigma_{\pi^{-1}(1)}(1)}, \dots, k^{\pi^{-1}(1)}_{\sigma_{\pi^{-1}(1)}(n)}; \sigma_{\pi^{-1}(1)}), \dots, (k^{\pi^{-1}(m)}_{\sigma_{\pi^{-1}(m)}(1)}, \dots, k^{\pi^{-1}(m)}_{\sigma_{\pi^{-1}(m)}(n)}; \sigma_{\pi^{-1}(m)}); \pi\big) \in (K \wr n) \wr m \end{aligned}$$

where $h'_{\pi^{-1}(i)}h_j = h_{\sigma_{\pi^{-1}(i)}(j)}k_j^{\pi^{-1}(i)}$, i.e. $h'_ih_j = h_{\sigma_i(j)}k^i_j$. Then composed with i,

$$g \mapsto \big(k^{\pi^{-1}(1)}_{\sigma_{\pi^{-1}(1)}(1)}, \dots, k^{\pi^{-1}(1)}_{\sigma_{\pi^{-1}(1)}(n)}, \dots, k^{\pi^{-1}(m)}_{\sigma_{\pi^{-1}(m)}(1)}, \dots, k^{\pi^{-1}(m)}_{\sigma_{\pi^{-1}(m)}(n)}; G'(\sigma_{\pi^{-1}(1)}, \dots, \sigma_{\pi^{-1}(m)}; \pi)\big) \in K \wr mn.$$

On the other hand, under $i_G^{K\wr mn}$,

$$\begin{aligned} g &\mapsto \psi(l_1^1, \dots, l_n^1, l_1^2, \dots, l_n^m) \\ &= (l^{\psi_1^{-1}(1)}_{\psi_2^{-1}(1)}, \dots, l^{\psi_1^{-1}(m)}_{\psi_2^{-1}(n)}; \psi) \end{aligned}$$

where $l^i_j \in K$ is the $(j + n(i-1))$th entry, we set $gg_ih_j = g_{\psi_1(i)}h_{\psi_2(j)}l^i_j$, and $\psi = (\psi_1, \psi_2) \in S_{(m,n)}$ is identified via $\psi(j + (i-1)n) = \psi_2(j) + (\psi_1(i) - 1)n$ following the enumeration. However, we also have

$$gg_ih_j = g_{\pi(i)}h_{\sigma_i(j)}k^i_j,$$

hence $k^i_j = l^i_j$, and $(\psi_1(i), \psi_2(j)) = (\pi(i), \sigma_i(j))$. It follows that

$$\begin{aligned} \psi(j + (i-1)n) &= \psi_2(j) + (\psi_1(i) - 1)n \\ &= \sigma_i(j) + (\pi(i) - 1)n \\ &= \sigma_{\pi(\pi^{-1}(i))}(j) + n(\pi(i) - 1) \\ &= G'(\sigma_{\pi^{-1}(1)}, \dots, \sigma_{\pi^{-1}(m)}; \pi)(j + (i-1)n) \end{aligned}$$

Thus, the permutations are the same. Furthermore,

$$(\psi_1, \psi_2)^{-1}(i,j) = \big(\pi^{-1}(i), \sigma^{-1}_{\pi^{-1}(i)}(j)\big)$$

so it follows that the K-elements in the two terms match, and the diagram commutes as desired. □

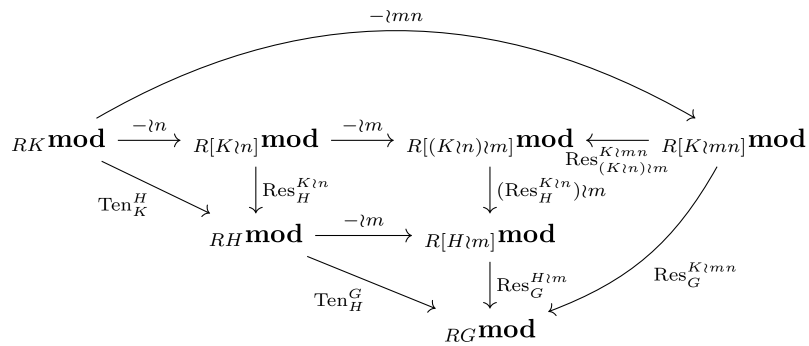
Proposition 1.4. Let $K \le H \le G$ be finite groups, with $[H : K] = n$ and $[G : H] = m$. Let M be a RK-module. Then,

$$\mathrm{Ten}_H^G \mathrm{Ten}_K^H M \cong \mathrm{Ten}_K^G M \quad \text{and} \quad (M \wr n) \wr m \cong \mathrm{Res}^{K\wr mn}_{(K\wr n)\wr m} M \wr nm.$$

Proof. By definition,

$$\operatorname{Ten}_H^G \operatorname{Ten}_K^H M = \operatorname{Res}_G^{H\wr m}\left(\operatorname{Res}_H^{K\wr n}(M \wr n) \wr m\right)$$

and $\operatorname{Ten}_K^G M = \operatorname{Res}_G^{K\wr mn}(M \wr mn)$. To prove these are isomorphic it suffices to show the following diagram commutes up to isomorphism:



Here, all restrictions are induced by the inclusions in the previous proposition, and Ten_K^G is the composite of the two outer curved arrows. In fact, all subdiagrams in the diagram except for the top commute precisely, not only up to isomorphism. The middle square commutes by definition of $(\operatorname{Res}_H^{K\wr n})\wr m$, as the lower composite corresponds to first applying restriction, then tensoring m times, while the upper composite corresponds to first tensoring m times then applying the same restriction to each of the m copies. The triangles containing Ten_K^H and Ten_H^G commute by definition. The rightmost triangle commutes by the commutativity of the inclusions proven in the previous proposition. Hence, it suffices to show the topmost diagram commutes up to isomorphism, which will prove both statements.

We construct the isomorphism as follows. Let V be a RK-module, and $(v_1^1 \otimes v_n^1) \otimes (v_1^2 \otimes \cdots \otimes v_n^m) \in (V \wr n) \wr m$. The mapping is induced as follows:

$$(v_1^1 \otimes \cdots \otimes v_n^1) \otimes (v_1^2 \otimes \cdots \otimes v_n^m) \mapsto v_1^1 \otimes \cdots \otimes v_n^1 \otimes v_1^2 \otimes \cdots \otimes v_n^m \in \operatorname{Res}_{(K\wr n)\wr m}^{K\wr mn} M \wr nm.$$

This obviously induces a bijective map which is $R[H^n]$-linear. To verify it is a module isomorphism, it suffices to verify the S_{mn}-actions are compatible, but this follows by the construction of G' arising from the enumeration in the previous proposition. □

Proposition 1.5. Let $H \leq G$ be finite groups and M a finitely generated RH-module which is projective as R-module. Then $M^* \wr n \cong (M \wr n)^*$ naturally for any $n \in \mathbb{N}$. In particular, we have a natural isomorphism $\operatorname{Ten}_H^G M^* \cong (\operatorname{Ten}_H^G M)^*$.

Proof. We have a natural (in all components) transformation of additive functors ${}_R\mathbf{mod}^{\times n} \to {}_R\mathbf{mod}$,

$$M_1^* \otimes_R \cdots \otimes_R M_n^* \mapsto (M_1 \otimes_R \cdots \otimes_R M_n)^*, \quad f_1 \otimes \cdots \otimes f_n \mapsto \left(m_1 \otimes \cdots \otimes m_n \mapsto f_1(m_1) \cdots f_n(m_n)\right)$$

It is easy to check that if all M_i are free R-modules, then it is a natural isomorphism, hence it follows that if all the M_i are projective R-modules, it is a natural isomorphism as well. Thus $M^* \wr n \cong (M \wr n)^*$ as R-modules.

It remains to verify the natural isomorphism is $H \wr n$-linear. Let $(h_1, \ldots, h_n; \pi) \in H \wr n$, then we compute:

$$\begin{aligned} \phi : (h_1, \ldots, h_n; \pi) \cdot f_1 \otimes \cdots \otimes f_n &= f_{\pi^{-1}(1)}(h_1^{-1} \cdot -) \otimes \cdots \otimes f_{\pi^{-1}(n)}(h_n^{-1} \cdot -) \\ &\mapsto \left(v_1 \otimes \cdots \otimes v_n \mapsto f_{\pi^{-1}(1)}(h_1^{-1} \cdot v_1) \cdots f_{\pi^{-1}(n)}(h_n^{-1} \cdot v_n)\right) \end{aligned}$$

On the other hand,

$$\begin{aligned}(h_1, \ldots, h_n; \pi) \cdot \phi : f_1 \otimes \cdots \otimes f_n & \\ &\mapsto (h_1, \ldots, h_n; \pi) \cdot \big(v_1 \otimes \cdots \otimes v_n \mapsto f_1(v_1) \cdots f_n(v_n)\big) \\ &= \big(v_1 \otimes \cdots \otimes v_n \mapsto f_1(h_{\pi(1)}^{-1} \cdot v_{\pi(1)}) \cdots f_n(h_{\pi(n)}^{-1} v_{\pi(n)})\big)\end{aligned}$$

noting that $(h_1, \cdots, h_n; \pi)^{-1} = (h_{\pi(1)}^{-1}, \cdots, h_{\pi(n)}; \pi^{-1})$. But these are the same functions ordered differently, so we conclude the natural isomorphism is $H \wr n$-linear, as desired. Since restriction commutes with duals, the last statement follows immediately. □

2 Tensor induction of chain complexes

The construction used for tensor induction of chain complexes appears to have first been constructed by Evens in 1961 in the construction of the Evens norm map, an analogous norm map to tensor induction for cohomology. While this construction has been used on occasion since then, the properties of this construction do not appear to have been studied in further detail, or at minimum have not been documented. We will study some basic properties of the construction, proving that it has all the analogous properties of tensor induction on modules and G-sets.

Definition 2.1. Let $H \leq G$ be finite groups with $[G : H] = n$. If $C = \cdots \to C_i \xrightarrow{d_i} C_{i-1} \to \cdots$ is a chain complex of RH-modules, then $C \otimes_R \cdots \otimes_R C = C^{\otimes n}$ is a $R[H^n]$-chain complex by diagonal action. Note that the transition maps are as follows:

$$\begin{aligned}d_{a_1,\ldots,a_n} : C_{a_1} \otimes_R \cdots \otimes_R C_{a_n} &\to (C \wr n)_{a_1+\cdots+a_n-1} \\ m_1 \otimes \cdots \otimes m_n &\mapsto \sum_{i=1}^{n} (-1)^{a_1+\cdots+a_{i-1}} m_1 \otimes \cdots \otimes d_{a_i}(m_i) \otimes \cdots \otimes m_n\end{aligned}$$

Define $C \wr n$ as a chain complex of $R[H \wr n]$-modules as follows: let $C \wr n$ be $C^{\otimes n}$ as chain complexes of $R[H \wr n]$-modules, and for $m_1 \otimes \cdots \otimes m_n \in C_{a_1} \otimes_R \cdots \otimes_R C_{a_n} \subseteq (C \wr n)_{a_1+\cdots+a_n}$ and $(h_1, \ldots, h_n; \pi) \in H \wr n$, then

$$\begin{aligned}(h_1, \cdots h_n; \pi) \cdot (m_1 \otimes \cdots \otimes m_n) &= (-1)^{\nu_\pi} h_1 m_{\pi^{-1}(1)} \otimes \cdots \otimes h_n m_{\pi^{-1}(n)} \\ &\in C_{a_{\pi^{-1}(1)}} \otimes_R \cdots \otimes_R C_{a_{\pi^{-1}(n)}} \subseteq (C \wr n)_{a_1+\cdots+a_n}\end{aligned}$$

where

$$\nu_\pi = \sum_{\substack{j<k \\ \pi(j)>\pi(k)}} a_j a_k$$

Denote by $\mathrm{Ten}_H^G(C)$ the restriction of $C \wr n$ from $H \wr n$ to G via the inclusion $G \hookrightarrow H \wr n$ described prior. In particular, the G-action is as follows:

$$g \cdot (m_1 \otimes \cdots \otimes m_n) = (-1)^{\nu_{\pi_g}} h_{\pi_g^{-1}(1)} m_{\pi_g^{-1}(1)} \otimes \cdots \otimes h_{\pi_g^{-1}(n)} m_{\pi_g^{-1}(n)}$$

This is the *tensor induced chain complex obtained from* C.

The sign change given by ν_π corresponds to writing π as a product of simple transpositions $(1, i)$ and for each one multiplying by a sign of $(-1)^{a_1 a_i}$. The sign change is necessary so that the transition maps are compatible with the G-action. We now prove the following analogs of the previously stated properties of tensor induction for modules or G-sets.

Proposition 2.2. Let $H \leq G$ be finite groups and C, D bounded chain complexes of RH-modules. Then for any $n \in \mathbb{N}$,

$$(C \wr n) \otimes_R (D \wr n) \cong (C \otimes_R D) \wr n.$$

In particular,

$$\mathrm{Ten}_H^G(C) \otimes_R \mathrm{Ten}_H^G(D) \cong \mathrm{Ten}_H^G(C \otimes_R D).$$

Proof. The latter statement follows from the former by applying restriction. Let $C = \cdots \to C_i \xrightarrow{d_i} C_{i-1} \to \cdots$ and $D = \cdots \to D_i \xrightarrow{e_i} D_{i-1} \to \cdots$. We claim the component-level maps

$$\phi : (C_{a_1} \otimes_R \cdots \otimes_R C_{a_n}) \otimes_R (D_{b_1} \otimes_R \cdots \otimes_R D_{b_n}) \cong (C_{a_1} \otimes_R D_{b_1}) \otimes_R \cdots \otimes_R (C_{a_n} \otimes_R D_{b_n}),$$

$$(c_1 \otimes \cdots \otimes c_n) \otimes (d_1 \otimes \cdots \otimes d_n) \mapsto (-1)^s (c_1 \otimes d_1) \otimes \cdots \otimes (c_n \otimes d_n),$$

where

$$s = \sum_{i=1}^{n} a_i \left(\sum_{j=1}^{i-1} b_j \right)$$

induce a chain complex isomorphism of $R[H \wr n]$-modules. It is straightforward to see this is $R[H^n]$-linear and bijective, and that it respects the S_n-action and differentials up to a sign. It remains to show that the signs are compatible with the S_n-action and the graded differentials.

We first verify the sign is compatible with the sign induced by the S_n action. It suffices to prove this for simple transpositions of the form $(k, k+1)$, and compare signs of of $-1\nu^C_{(k,k+1)} + \nu^D_{(k,k+1)} + s'$ (corresponding to applying ϕ first) and $(-1)^{s+\nu^{C\otimes D}_{(k,k+1)}}$ (corresponding to first permuting), where s' corresponds to the sign calculation after permuting,

$$s' = \sum_{i=1}^{n} a_{(k,k+1)(i)} \left(\sum_{j=1}^{i-1} b_{(k,k+1)(j)} \right).$$

It follows that $\nu^C_{(k,k+1)} + \nu^D_{(k,k+1)} = a_k a_{k+1} + b_k b_{k+1}$ and $\nu^{C\otimes D}_{(k,k+1)} = (a_k + b_k)(a_{k+1} + b_{k+1})$. Moreover, it is routine to compute that $s' - s = a_i b_{k+1} - a_{k+1} b_i$, so we observe

$$s' + \nu^C_{(k,k+1)} + \nu^D_{(k,k+1)} - (s + \nu^{C\otimes D}_{(k,k+1)}) = -2a_{k+1} b_i,$$

hence the signs match, as desired. Thus, the choice of s produces isomorphisms ϕ compatible with the $R[H \wr n]$-module structure.

We now verify the choice of s commutes with the graded differentials. Consider the differential $d^C_{a_i}$ coming from the complex C in the ith component. Following ϕ first, then the differential yields the sign

$$(-1)^{s+a_1+b_1+\cdots+a_{i-1}+b_{i-1}}.$$

On the other hand, first following the differential, then ϕ, yields the sign

$$(-1)^{a_1+\cdots+a_{i-1}+s-(b_1+\cdots+b_{i-1})}.$$

The exponents differ only by signs, hence they have the same parity, so the isomorphism commutes with all C-differentials. Similarly, if we consider the differentials $d^D_{b_j}$ coming from the complex D int he jth component, if we follow ϕ first, then the differential, we obtain the sign

$$(-1)^{s+a_1+b_1+\cdots+a_{j-1}+b_{j-1}+a_j}.$$

If we follow the differential, then ϕ, we obtain

$$(-1)^{a_1+\cdots+a_n+b_1+\cdots+b_{j-1}+s-(a_{j+1}+\cdots+a_n)}.$$

The exponents match, so we conclude ϕ commutes with all differentials, as desired. □

Proposition 2.3. Let $H \leq G$ be finite groups, and $C = \cdots \to C_i \xrightarrow{d_i} C_{i-1} \to \cdots$ a chain complex of RH-modules which are projective as R-modules. Then $C^* \wr n \cong (C \wr n)^*$, where C^* denotes the chain complex induced by the dual. In particular, $\mathrm{Ten}_H^G(C^*) \cong (\mathrm{Ten}_H^G C)^*$.

Proof. We have a natural (in all components) transformation of additive functors ${}_R\mathbf{mod}^{\times n} \to {}_R\mathbf{mod}$,

$$M_1^* \otimes_R \cdots \otimes_R M_n^* \mapsto (M_1 \otimes_R \cdots \otimes_R M_n)^*, \quad f_1 \otimes \cdots \otimes f_n \mapsto \big(m_1 \otimes \cdots \otimes m_n \mapsto f_1(m_1) \cdots f_n(m_n)\big)$$

It is easy to check that if all M_i are free R-modules, then it is a natural isomorphism, hence it follows that if all the M_i are projective R-modules, it is a natural isomorphism as well. From this, we obtain componentwise natural isomorphisms

$$C_{a_1}^* \otimes_R \cdots \otimes_R C_{a_n}^* \cong (C_{a_1} \otimes_R \cdots \otimes_R C_{a_n})^*.$$

It remains to verify the componentwise isomorphisms are $H \wr n$-linear. Let $(h_1, \ldots, h_n; \pi) \in H \wr n$, then we compute:

$$\begin{aligned} \phi : (h_1, \ldots, h_n; \pi) \cdot f_1 \otimes \cdots \otimes f_n &= f_{\pi^{-1}(1)}(h_1^{-1} \cdot -) \otimes \cdots \otimes f_{\pi^{-1}(n)}(h_n^{-1} \cdot -) \\ &\mapsto \big(v_1 \otimes \cdots \otimes v_n \mapsto f_{\pi^{-1}(1)}(h_1^{-1} \cdot v_1) \cdots f_{\pi^{-1}(n)}(h_n^{-1} \cdot v_n)\big) \end{aligned}$$

On the other hand,

$$\begin{aligned} (h_1, \ldots, h_n; \pi) \cdot \phi : f_1 \otimes \cdots \otimes f_n &\mapsto (h_1, \ldots, h_n; \pi) \cdot \big(v_1 \otimes \cdots \otimes v_n \mapsto f_1(v_1) \cdots f_n(v_n)\big) \\ &= \big(v_1 \otimes \cdots \otimes v_n \mapsto f_1(h_{\pi(1)}^{-1} \cdot v_{\pi(1)}) \cdots f_n(h_{\pi(n)}^{-1} v_{\pi(n)})\big) \end{aligned}$$

noting that $(h_1, \cdots, h_n; \pi)^{-1} = (h_{\pi(1)}^{-1}, \cdots, h_{\pi(n)}; \pi^{-1})$ and that the resulting functions belong to $(C_{\pi^{-1}(1)}) \otimes C_{\pi^{-1}(n)})^*$. But these are the same functions ordered differently, so we conclude the componentwise natural isomorphisms are $H \wr n$-linear. It remains to verify the maps are compatible with the graded differential structure. By naturality, the following diagram commutes, and the result follows.

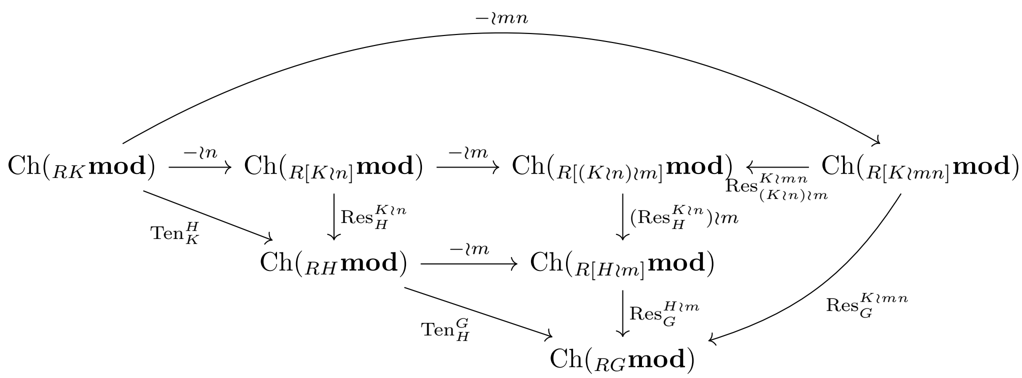
$$\begin{array}{ccc} C_{a_1}^* \otimes_R \cdots \otimes_R C_{a_n}^* & \xrightarrow{\quad\phi\quad} & (C_{a_1} \otimes_R \cdots \otimes_R C_{a_n})^* \\ {\scriptstyle (-1)^{-(a_1 + \cdots + a_{i-1})}\, \mathrm{id} \otimes \cdots \otimes (d_{a_i+1}^i)^* \otimes \cdots \otimes \mathrm{id}} \downarrow & & \downarrow {\scriptstyle (-1)^{a_1 + \cdots + a_{i-1}} (\mathrm{id} \otimes \cdots \otimes d_{a_i+1}^i \otimes \cdots \otimes \mathrm{id})^*} \\ C_{a_1}^* \otimes_R \cdots \otimes_R C_{a_i+1}^* \otimes_R \cdots \otimes_R C_n & \xrightarrow{\phi} & (C_{a_1} \otimes_R \cdots \otimes_R C_{a_i+1} \otimes_R \cdots \otimes_R C_{a_n})^* \end{array}$$

□

Proposition 2.4. Let $K \leq H \leq G$ be finite groups with $[H : K] = n$ and $[G : H] = m$. Let C be a chain complex of RK-modules. Then,

$$\mathrm{Ten}_H^G \mathrm{Ten}_K^H C \cong \mathrm{Ten}_K^G C \quad \text{and} \quad (C \wr n) \wr m \cong C \wr nm.$$

Proof. It again suffices to prove that the following diagram commutes:



The four bottom-most subdiagrams commute for the same reasons as before, therefore it only remains to show $(C \wr n) \wr m \cong \mathrm{Res}^{K\wr mn}_G C \wr nm$. Let $C = \cdots \to C_i \xrightarrow{d_i} C_{i-1} \to \cdots$, then we have a componentwise isomorphism,

$$\begin{aligned}&(C_{a_{1,1}} \otimes_R \cdots \otimes_R C_{a_{n,1}}) \otimes_R \cdots \otimes_R (C_{a_{1,m}} \otimes_R \cdots \otimes_R C_{a_{n,m}})\\ &\to C_{a_{1,1}} \otimes_R \cdots \otimes_R C_{a_{n,1}} \otimes_R \cdots \otimes_R C_{a_{1,m}} \otimes_R \cdots \otimes_R C_{a_{n,m}}\end{aligned}$$

which is constructed similarly to the module-theoretic version of the theorem. It follows that this isomorphism is compatible with the graded differential structure (no sign changes are necessary) and is $R[(K \wr n) \wr m]$-linear. □

3 Restriction to trivial source and linear source

It is known that tensor induction of modules restricts to a functor of chain complexes of p-permutation modules. However, in general, tensor induction does not restrict to a functor of p-permutation modules, as the following example demonstrates:

Example 3.1. Let $C \in \mathrm{Ch}^b({}_{\mathcal{O}}\mathbf{mod})$ be the contractible chain complex $0 \to \mathcal{O} \to \mathcal{O} \to 0$, with the last nonzero term in degree 0. Then,

$$C \wr 2 = 0 \to \mathcal{O}^- \to \mathcal{O}C_2 \to \mathcal{O},$$

where $\mathcal{O}^-$ corresponds to the sign representation sending the nontrivial element of C_2 to -1.

However, when working over any field in a p-modular system, tensor induction indeed restricts to a functor of chain complexes of p-permutation modules. This is obvious over K since every module is a p-permutation module. Additionally, over $\mathcal{O}$, tensor induction restricts to a functor of chain complexes of *linear* source modules.

Theorem 3.2. Let H be a finite group, n a natural number, $(K, \mathcal{O}, k)$ a p-modular system.

(a) $-\wr n : \mathrm{Ch}^b({}_{kH}\mathbf{mod}) \to \mathrm{Ch}^b({}_{k[H\wr n]}\mathbf{mod})$ restricts to a functor $-\wr n : \mathrm{Ch}^b({}_{kH}\mathbf{triv}) \to \mathrm{Ch}^b({}_{k[H\wr n]}\mathbf{triv})$.

(b) $-\wr n : \mathrm{Ch}^b({}_{\mathcal{O}H}\mathbf{mod}) \to \mathrm{Ch}^b({}_{\mathcal{O}[H\wr n]}\mathbf{mod})$ restricts to a functor $-\wr n : \mathrm{Ch}^b({}_{\mathcal{O}H}\mathbf{lin}) \to \mathrm{Ch}^b({}_{\mathcal{O}[H\wr n]}\mathbf{lin})$.

Lemma 3.3. Let k be any field and G a finite group. Any kG-module M with k-dimension 1 is a p-permutation module.

Proof. If F has characteristic 0, then all modules are trivial source modules so this is obvious. Otherwise assume F has characteristic p. Consider any element $g \in G$ with order a power of p. Since k has no primitive

pth roots of unity, g acts trivially on M, hence after restriction to a p-subgroup $P \le G$, $\operatorname{Res}^G_P M$ is the trivial representation. Since a module is p-permutation module if and only if upon restriction to all P-subgroups it is a permutation module, the result follows. $\square$

Proof of theorem. Let R denote either $\mathcal{O}$ or k, and let $C \in \operatorname{Ch}^b({}_{RH}\mathbf{mod})$ be a bounded complex of finitely generated p-permutation modules and denote the degree i component by C_i. Assume without loss of generality that $C_i = 0$ for $i \le 0$. When considered as a chain complex of $R[H^n]$-modules, each direct summand of $(C \wr n)_k$ is of the form $C_{a_1} \otimes_k \cdots \otimes_k C_{a_n}$, where $\sum_{i=1}^n a_i = l$. Then, S_n acts on the $R[H^n]$-direct summands of $(C \wr n)_l$ as follows:

$$^{\pi}(C_{a_1} \otimes_k \cdots \otimes_k C_{a_n}) := C_{a_{\pi^{-1}(1)}} \otimes_k \cdots \otimes_k C_{a_{\pi^{-1}(n)}},$$

corresponding to the S_n-action defined on $C \wr n$. Fix $a_1, \dots, a_n$, then we define the $R[H^n]$-module $M_{a_1,\dots,a_n}$ as follows. Set $C_{a_1,\dots,a_n} := C_{a_1} \otimes_k \cdots \otimes_k C_{a_n}$, then

$$M_{a_1,\dots,a_n} = \bigoplus_{M' \in S_n \cdot C_{a_1,\dots,a_n}} M',$$

i.e. $M_{a_1,\dots,a_n}$ is the direct sum of the S_n-orbit of $C_{a_1,\dots,a_n}$. It follows via the construction that $M_{a_1,\dots,a_n}$ is a $R[H \wr n]$-module, and

$$(C \wr n)_l = \bigoplus_{a_1,\dots,a_n \text{ is a partition of } l} M_{a_1,\dots,a_n}.$$

(a) It suffices to show if $C \in \operatorname{Ch}^b({}_{kH}\mathbf{triv})$, then for any choice of $a_1, \dots, a_n$, $M_{a_1,\dots,a_n} \in {}_{k[H \wr n]}\mathbf{triv}$. Since all $C_i \in {}_{kH}\mathbf{triv}$, then $C_1 \oplus \cdots \oplus C_m$ is a direct summand of some $N \in {}_{kH}\mathbf{perm}$. Then as $k[H^n]$-modules, any module living in the S_n-orbit of $C_{a_1,\dots,a_n}$ is a direct summand of $N \otimes_k \cdots \otimes_k N$, and thus as $k[H \wr n]$-modules, $M_{a_1,\dots,a_n}$ is a direct summand of $N \wr n$. However, for $K \le H$ a vertex of N, we have an isomorphism $N \cong \operatorname{Ind}^H_K(k)$, and under this identification, an isomorphism of $k[H \wr n]$-modules:

$$N \wr n \cong \operatorname{Ind}^{H \wr n}_{K \wr n}(k \wr n)$$
$$(h_1 \otimes v_1) \otimes \cdots \otimes (h_n \otimes v_n) \mapsto (h_1, \cdots, h_n; \mathrm{id}) \otimes (v_1 \otimes \cdots \otimes v_n)$$

Here, $k \wr n$ has k-dimension 1 and has $K \wr n$-action given by

$$\pi(k_1, \dots, k_n)(v_1 \otimes \cdots \otimes v_n) = (-1)^{\nu_\pi}(v_{\pi^{-1}(1)} \otimes \cdots \otimes v_{\pi^{-1}(n)}).$$

By the previous lemma, $k \wr n$ is a trivial source module, hence $k \wr n$ is a direct summand of $\operatorname{Ind}^{K \wr n}_L k$ for some $L \le K \wr n$. Hence $M_{a_1,\dots,a_n}$ is also a direct summand of $\operatorname{Ind}^{H \wr n}_{K \wr n} \operatorname{Ind}^{K \wr n}_L k = \operatorname{Ind}^{H \wr n}_L k$, thus is a p-permutation module.

(b) The proof follows similarly as before with $\mathcal{O}$ in place of k, except the isomorphism $N \wr n \cong \operatorname{Ind}^{H \wr n}_{K \wr n}(\mathcal{O} \wr n)$ is sufficient to demonstrate $M_{a_1,\dots,a_n}$ is a linear source module, since $\mathcal{O} \wr n$ as described earlier has $\mathcal{O}$-rank 1.

$\square$