

STATEMENT OF RESEARCH

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My research involves representation theory, tensor-triangular geometry, and equivariant topology, with a primary goal of understanding group- and representation-theoretic phenomena via techniques from all these realms. The derived category of permutation modules considered by Balmer–Gallauer [6, 7, 8, 9, 10] has been a central theme of my research. In particular, I’ve leveraged the invertible objects of the category, endotrivial complexes (which I classified), [68] to gain new insight into the category’s geometric and cohomological properties [70] and understand group representation-theoretic phenomena through an equivariant lens. Ideas, objects, and techniques arising in geometry and equivariant topology have all been necessary to develop our understanding of these phenomena.

Here, unlike the situation for the derived or stable module categories, one cannot simply rely on ordinary group cohomology to fully understand the geometry. There is p -local cohomological phenomena as well, and both detecting it and understanding how it is glued together is achieved via techniques and constructions involving endotrivial complexes; as a result, one can use these invertible objects to give new understanding to ordinary cohomology. Though we are now close to achieving my main goal of the past two years, constructing and understanding an accessible twisted cohomology theory which completely controls the geometry of permutation modules, [44, 70] numerous major questions remain.

Moreover, the techniques I’ve developed in this setting are already being adopted elsewhere: in ongoing work, Benson–Iyengar–Krause–Pevtsova are leveraging permutation twisted cohomology via Schur–Weyl duality to describe the tensor-triangular geometry of Schur algebras.

Beyond permutation modules, I have also been actively working to develop the tensor-triangular geometry of other representation theoretic categories of interest, [47, 72] such as the permutation and representation categories of oligomorphic groups, infinite groups arising from model theory, introduced by Harman–Snowden. [49] I’ve also had a hand in developing the theory of noncommutative tensor-triangular geometry introduced by Nakano–Vashaw–Yakimov, [34, 72] and am currently working to deduce the geometry of familiar noncommutative representation-theoretic examples, such as categories of bimodules of group or block algebras. Some of these noncommutative developments have been through the lens of frame theory, and an ongoing project is to describe and compare different support theories for triangulated categories in this way. Even beyond these projects, I have other projects concerning representation spheres, global Mackey functors, and Galois descent in block theory.

1. THE GEOMETRY OF PERMUTATION MODULES

1.1. Tensor-triangular geometry. We begin with a brief introduction to tensor-triangular geometry. Tensor-triangular geometry (tt-geometry for short) is the study of tensor-triangulated categories, i.e., triangulated categories with a compatible symmetric monoidal structure, via algebro-geometric methods. To any essentially small tensor-triangulated category $\mathcal{K}$, one associates its Balmer spectrum $\mathrm{Spc}(\mathcal{K})$, a topological space, with support data for every object $x \in \mathcal{K}$. This forms, in a precise way, the *universal support* of $\mathcal{K}$. Moreover, $\mathrm{Spc}(\mathcal{K})$ classifies the (radical) thick tensor-ideals of $\mathcal{K}$; in particular, it determines when one object of $\mathcal{K}$ can be built from another via natural tensor-triangular operations. This can be leveraged to study nilpotence, descent, [4, 5] and classifications for ‘big’ categories. [12, 16] See [3, 84] for in-depth overviews.

Given a tensor-triangulated category $\mathcal{K}$, the first question is to deduce $\mathrm{Spc}(\mathcal{K})$. Doing so often recovers already well-understood spaces. For instance, if G is a finite group (scheme) and k a field of prime characteristic p , then the Balmer spectrum of the bounded derived category of kG -modules, $\mathrm{Spc}(\mathrm{D}_b(\mathrm{mod}(kG)))$, is nothing more than $\mathrm{Spec}^h(H^\bullet(G; k))$, Carlson’s (homogeneous) cohomological support variety, [28] and the Balmer spectrum of the stable module category $\mathrm{Spc}(\mathrm{stmod}(kG))$ is correspondingly the projectivization, $\mathrm{Proj}(H^\bullet(G; k))$. [19, 39] If X is a quasi-compact quasi-separated scheme (e.g., $\mathrm{Spec}(R)$), then $\mathrm{Spc}(\mathrm{D}_{\mathrm{perf}}(X))$ recovers X viewed as a topological space [55, 80, 85]. Finally, the Balmer spectrum of SH^c , the category of (compact) genuine spectra recovers the so-called ‘chromatic tower’ related to Ravenel’s conjectures in chromatic homotopy theory. [33, 81]

1.2. Permutation modules. Let us turn to permutation modules. We let G be a finite group and k a field of positive characteristic p . The representation theory of kG is almost always wild, that is, classifying the indecomposable modules in full is a hopeless endeavor. [36] On the other hand, permutation kG -modules, i.e., those admitting a G -stable k -basis, and their direct summands, p -permutation modules, are a natural, easily-described collection of objects. Explicitly, every permutation module is a direct sum of transitive permutation modules of the form $k(G/H)$.

Despite their innocuous nature, Balmer–Gallauer deduced a shocking result: permutation modules recover the entire wild representation theory of kG .

Theorem. [7] *Every finitely generated kG -module admits a finite resolution by p -permutation modules. Consequently, the tt-category $\mathcal{K}(G) = \mathcal{K}(G; k) := K_b(\text{perm}(G; k)^{\natural})$, the bounded homotopy category of p -permutation modules, admits $D_b(\text{mod}(kG))$ as a Verdier quotient.*

Geometrically, $\text{Spc}(D_b(\text{mod}(kG)))$ openly embeds into $\text{Spc}(\mathcal{K}(G))$. Therefore, one may ask what the other ‘piece’ of the spectrum is. In fact, the Balmer spectrum $\text{Spc}(\mathcal{K}(G; k))$ is built as a set from ‘ p -local’ cohomology:

$$\text{Spc}(\mathcal{K}(G)) = \bigsqcup_{H \in \text{Sub}_p(G)/G} \text{Spc}(D_b(\text{mod}(k(N_G(H)/H)))) \cong \bigsqcup_{H \in \text{Sub}_p(G)/G} \text{Spec}^h(H^*(N_G(H)/H)).$$

Morally, one can think of complexes of permutation modules as being built of ‘ p -local cohomological data,’ a perspective adopted by Broué’s abelian defect group conjecture.

In the spirit of tt-geometry extending beyond the realm of any individual field, this permutation module category is not only of interest to representation theorists. The category $\mathcal{K}(G)$ is the compact part of the (big) *derived category of permutation modules* $\mathcal{D}(G) = \mathcal{D}(G; k) := D\text{Perm}(G; k)$, which admits equivalences to categories of cohomological Mackey functors, Artin motives, and modules in equivariant spectra. [8, 40]

The question of determining the topology of $\text{Spc}(\mathcal{K}(G))$, i.e., how the p -local cohomological data is glued together, is subtler. Balmer–Gallauer’s deduction [9] has two components: first, a colimit theorem reducing to the case of G elementary abelian. Then, with G elementary abelian, the authors constructed a *permutation twisted cohomology ring* $H^{\bullet\bullet}(G)$ analogous to the construction of twisted cohomology in algebraic geometry.

In fact, $H^{\bullet\bullet}(G)$ can also be viewed as a natural topological object: it is the ‘effective cone’ of the $\text{RO}(G)$ -graded Bredon homology ring, a view considered by Hausmann–Schwede. [53] Explicitly, the twisted cohomology ring $H^{\bullet\bullet}(G)$ is built of morphisms from the tensor unit to ‘twists,’ certain invertible objects of $\mathcal{K}(G)$, termed *endotrivial complexes*, that is, chain complexes C for which $C^* \otimes_k C$ is homotopy equivalent to the trivial kG -module.

Then, for G an elementary abelian p -group, Balmer–Gallauer prove the Balmer spectrum $\text{Spc}(\mathcal{K}(G))$ is openly immersed in the homogeneous spectrum of $H^{\bullet\bullet}(G)$. The role of endotrivial complexes here is significant: unlike the case of $D_b(\text{mod}(kG))$, where the cohomology ring (the graded endomorphism ring of the tensor unit) recovers the Balmer spectrum in full, the cohomology ring of $\mathcal{K}(G)$ provides essentially no information, as $\mathcal{K}(G)$ is a homotopy category.

Like in the setting of algebraic geometry, the endotrivial complexes used here are *tt-line bundles*: they are locally trivial up to shifting with respect to an open cover of $\text{Spc}(\mathcal{K}(G))$, and the local shifts detect new p -local cohomological data. In fact, $\text{Spc}(\mathcal{K}(G))$ is determined under this open cover; $H^{\bullet\bullet}(G)$ locally identifies with the usual cohomology ring, and the authors deduce homeomorphisms

$$U(H) = \text{Spc}(\mathcal{K}(G)|_{U(H)}) \xrightarrow{\sim} \text{Spec}^h(\text{End}_{\mathcal{K}(G)|_{U(H)}}^{\bullet}(\mathbb{1}))$$

and the topology of $\text{Spc}(\mathcal{K}(G))$ is built up from these local pieces. In particular, $\text{Spc}(\mathcal{K}(G))$ admits scheme structure in the sense of Hesselholt–Pstrągowski. [54]

Given the importance of endotrivial complexes in the elementary abelian case, the following question is natural: to what extent can the invertible objects, endotrivial complexes, inform our understanding of the tt-geometry of $\mathcal{K}(G)$ in greater generality?

1.3. Understanding endotriviality. To answer this question, we first need a better understanding of the endotriviality in general. In my dissertation, I deduced the structure of $\text{Pic}(\mathcal{K}(G))$, the abelian group of endotrivial complexes, for any finite group G and field k . [68] In joint work with Gómez, we extended the classification to endotriviality over a commutative Noetherian ring.

Informally, endotrivial complexes are almost completely determined by numerical data on prime power subgroups, which we call *h-marks*, virtual analogues of dimension functions of representation spheres. These functions are always *Borel-Smith* functions [35], functions arising both in topological and representation-theoretic contexts, including in Bouc’s classification of endopermutation modules. [23, 68, 86]

Theorem 1.1. [43, 68] *Let G be a finite group and R a connected commutative Noetherian ring. Then*

$$\mathrm{Pic}(\mathcal{K}(G; R)) = \mathrm{Pic}(R) \times \mathrm{CF}_b^{gl}(G; R) \times \mathcal{M}(G; R),$$

where $\mathrm{CF}_b^{gl}(G; R)$ is a group of tuples of Borel-Smith functions glued at the trivial subgroup and $\mathcal{M}(G; R) \leq \mathrm{Hom}(G, R^\times)$ is a subgroup consisting of all group homomorphisms $\varphi: G \rightarrow R^\times$ for which R_φ is a direct summand of a permutation module. (see [43] for an explicit description and extension to non-connected rings)

Unlike in the field case which was deduced purely representation-theoretically, [68] this deduction required the use of homotopy theory and abstract tensor-triangular geometric tools. One interesting descent-theoretic result the classification hinges on is as follows. One may enhance $\mathcal{K}(G)$ and $\mathcal{D}(G)$ to the level of stable ∞ -categories. From here, $\mathcal{K}(G)$ may be expressed as an ∞ -categorical limit via its prime power subgroups. This enables the study of permutation modules via descent-theoretic techniques often seen in topology. [63, 64]

Theorem 1.2. [43] *Let G be a finite group and R a commutative Noetherian ring. Then*

$$\mathcal{K}(G; R) \cong \lim_{G/H \in \mathcal{O}_\Phi(G)^{\mathrm{op}}} \mathcal{K}(H; R),$$

where $\mathcal{O}_\Phi(G)$ denotes the full subcategory of the orbit category of G with objects with isotropy in prime power subgroups.

While endotrivials can be described by numerical and representation theoretic data, and sometimes have topological models, explicitly ‘writing down’ an indecomposable representative for an isomorphism class of endotrivials is surprisingly difficult.

Question 1.3. *Given a Borel-Smith function h , find a (possibly computational) method of determining an explicit indecomposable endotrivial complex C with h -mark function h .*

Loosely speaking, endotrivial complexes are algebraic analogues of invertible genuine G -spectra. Endotrivials arise from them, in particular, representation spheres, 1-point compactifications of real representations, and we have a group homomorphism

$$\mathcal{B}: \mathrm{RO}(G) \rightarrow \mathrm{Pic}(\mathcal{K}(G; R))$$

If G is a p -group and k a field of characteristic p , this map is surjective, but not in general. [68, 43] The following belief complements the previous.

Conjecture 1.4. *Given a representation sphere S^V , there exists a ‘unique’ cellularization whose associated cellular chain complex is indecomposable in $\mathrm{Ch}_b(\mathrm{perm}(G; k))$.*

Recall the endotrivial twists of Balmer–Gallauer’s twisted cohomology ring are tt-line bundles. I proved in [70] that if G is a p -group and $R = k$ a field of characteristic p , all endotrivials are tt-line bundles; this was the first step towards extending twisted cohomology. For finite groups over a commutative Noetherian ring, with Gómez, we deduced that while not all endotrivials are line bundles, certain orientable endotrivials are.

Theorem 1.5. [43, 70] *Let G be a finite group, R a commutative Noetherian ring, and $\mathrm{RO}^+(G) \leq \mathrm{RO}(G)$ denote the subgroup of orientable representation spheres mod $\mathrm{char}(R)$. Any endotrivial in the image of $\mathcal{B}: \mathrm{RO}^+(G) \rightarrow \mathrm{Pic}(\mathcal{K}(G; R))$ is a tt-line bundle.*

To show this line bundle property, I constructed what I call *forerunners*, certain morphisms from the tensor unit into ‘effective’ endotrivials, those corresponding to oriented representation spheres. Forerunners link p -local cohomological data to the global setting, and predict quasi-isomorphisms as precisely as possible. Inverting them produces the desired open cover of $\mathrm{Spc}(\mathcal{K}(G))$.

Theorem 1.6. [43, 70] *Let G be a finite group and R a commutative Noetherian ring. Let C be an effective endotrivial which is orientable modulo $\mathrm{char}(R)$. For every prime p not invertible in R and p -subgroup $H \leq G$, there exists a chain*

homomorphism $\iota_C^H: R \rightarrow C[-h_C(H)]$ such that $\Psi_p^H(\iota_C^H)$ is a quasi-isomorphism, where Ψ_p^H denotes generalized modular fixed points at H .

Numerous questions remain about these homomorphisms; these are posed in [70]. For instance, the construction chooses a cellularization of S^V ; it is not clear if the morphism is independent of the choice of cellularization, but we expect this to be the case (up to scaling).

1.4. Geometric implications. It turns out that the effective endotrivial, and in particular, forerunners, play a surprisingly significant part in the geometry of permutation modules for all finite groups. I've achieved my main goal of the past year, extending twisted cohomology, although in general we still do not fully understand the ring outside of small cases. Again, these rings are built by considering morphisms from the tensor unit to the orientable endotrivial arising from representation spheres, which are tt-line bundles.

There are two theories, each interesting for their own reasons. The first, for p -groups, is the generalization of Balmer–Gallauer's ring to all finite groups, equivalently, the effective cone of the $\mathrm{RO}(G)$ -graded Bredon homology ring. [25, 53, 62]

Theorem 1.7. [44, 70] *Let G be a finite p -group. We construct a twisted cohomology ring $H^{\bullet\bullet}(G)$ with a continuous comparison map*

$$\mathrm{comp}_G: \mathrm{Spc}(\mathcal{K}(G)) \rightarrow \mathrm{Spec}^h(H^{\bullet\bullet}(G))$$

which is an open immersion. Moreover, there exists a quasi-compact open cover $\{U(H)\}$ of $\mathrm{Spc}(\mathcal{K}(G))$ indexed by conjugacy classes of p -subgroups of G and localizations $\mathcal{O}_G^\bullet(H)$ of $H^{\bullet\bullet}(G)$ such that the comparison map restricts to a homeomorphism

$$\mathrm{comp}_H: U(H) \xrightarrow{\sim} \mathrm{Spec}^h(\mathcal{O}_G^\bullet(H)).$$

The ring $H^{\bullet\bullet}(G)$ is Noetherian when G is a Dedekind p -group or isomorphic to D_8 . Finally, $H^{\bullet\bullet}(G)$ admits a twist-zero localization isomorphic to $H^\bullet(G)$.

Originally, that comp_G was an open immersion was conditional on $H^{\bullet\bullet}(G)$ being Noetherian; this restriction is removed in the forthcoming [44]. Unfortunately, $H^{\bullet\bullet}(G)$ is difficult to describe in general. Even showing Noetherianity for $G = D_8$ appears to not have been considered before, as presentations of the $\mathrm{RO}(G)$ -graded Bredon homology ring have only been determined in small cases. [42, 53]

Conjecture 1.8. *Noetherianity of $H^{\bullet\bullet}(G)$ holds for any p -group G .*

However, for finite groups, an amazing observation is that a single well-chosen endotrivial suffices to construct a bigraded cohomology ring whose spectrum contains $\mathrm{Spc}(\mathcal{K}(G))$. I defined in joint work in preparation with Gómez a ‘reduced’ bigraded version of the twisted cohomology ring under which the same results hold independent of Noetherianity. This ring is twisted only by the oriented endotrivial associated to the regular representation $\mathbb{R}G$. This endotrivial, although hard to describe explicitly, has highly desirable properties that allow it to ‘stratify’ the different p -subgroups of G .

Theorem 1.9. [44] *Let G be a finite group. We construct a ‘reduced’ twisted cohomology ring $H^{\bullet\bullet}(G)'$ with a continuous comparison map*

$$\mathrm{comp}_G: \mathrm{Spc}(\mathcal{K}(G)) \rightarrow \mathrm{Spec}^h(H^{\bullet\bullet}(G)')$$

which is an open immersion. Moreover, there exists a quasi-compact open cover $\{U(H)'\}$ of $\mathrm{Spc}(\mathcal{K}(G))$ indexed by conjugacy classes of p -subgroups of G and localizations $\mathcal{O}_G^\bullet(H)'$ of $H^{\bullet\bullet}(G)'$ such that the comparison map restricts to a homeomorphism

$$\mathrm{comp}_H: U(H)' \xrightarrow{\sim} \mathrm{Spec}^h(\mathcal{O}_G^\bullet(H)').$$

In particular, $(\mathrm{Spc}(\mathcal{K}(G)), \mathcal{O}_{\mathrm{Spc}(\mathcal{K}(G))})$ is a Dirac scheme in the sense of [54]. Finally, $H^{\bullet\bullet}(G)'$ admits a twist-zero localization isomorphic to $H^\bullet(G)$.

The question of actually describing either cohomology ring, computationally or otherwise, remains an intriguing challenge. We expect that while the reduced ring has better properties, it will actually be much harder to describe beyond small groups. To our knowledge, either case has only been worked out in small, abelian examples. [42, 53] We expect that generative AI tools may be quite useful for these computations.

Question 1.10. *Give presentations for the rings $H^{\bullet\bullet}(G)$ and $H^{\bullet\bullet}(G)'$ in explicit cases.*

However, the above question can be refined. We prove in both cases that the comparison map is an open immersion by first finding a finitely generated subring for which the theorem holds, then lifting the result to the full ring. Lifting to the full ring very roughly corresponds to blowing up the space outside of $\mathrm{Spc}(\mathcal{K}(G))$ in $\mathrm{Spec}^h(H^{\bullet\bullet}(G))$. The more interesting question is instead to determine a presentation for a more reasonable subring.

Question 1.11. *Find an explicit set of generators and relations for a subring of $H^{\bullet\bullet}(G)$ or $H^{\bullet\bullet}(G)'$ recovering the spectrum $\mathrm{Spc}(\mathcal{K}(G))$ with as little extraneous data as possible.*

We suspect that it is possible to choose these subrings such that one actually obtains $\mathrm{Spc}(\mathcal{K}(G))$ on the nose plus a single closed point. This belief has been verified by Benson–Iyengar–Krause–Pevtsova in their ongoing work regarding the tensor-triangular geometry of Schur algebras. Using Schur–Weyl duality, one can transport the questions they pose to the permutation-module-theoretic setting. The authors have adapted our techniques to their setting, although their analogous bigraded twisted cohomology ring has a slightly different twist. They’ve demonstrated that the techniques here can be extended beyond the setting of $\mathcal{K}(G)$; we are hopeful that there are yet more settings where similar techniques can be applied.

Additionally, in joint work with Gallauer, we utilized them to deduce the periodic locus of $\mathrm{Spc}(\mathcal{K}(G))$. One may think of this invariant as both a generalization of periodicity in groups and of motivic phenomena.

Theorem 1.12. [41] *Let G be a finite group. The periodic locus of $\mathrm{Spc}(\mathcal{K}(G))$ is the subset of non-closed points.*

However, the question of explicitly deducing the periods of each point remains open. In small examples, we’ve computed that for p -groups, the periods are in fact described in their entirety by endotrivial (equivariantly real representations, equivalently Borel–Smith functions) and specialization data in $\mathrm{Spc}(\mathcal{K}(G))$. We will investigate this in future work. For non- p -groups, this fails, for instance with the Mathieu group M_{11} .

Conjecture 1.13. *Let G be a p -group. The periodicity data of $\mathrm{Spc}(\mathcal{K}(G))$ can be described by endotrivial and specialization data alone.*

1.5. More questions. The following questions focus on Picard groups, and in particular, relating the derived category of permutation modules to other representation-theoretic and homotopy-theoretic phenomena.

- A question of significant interest in stable homotopy theory is the computation of ‘local’ Picard groups; one such question regarding $K(n)$ -local spectra was recently settled by Barthel–Schlank–Stapleton–Weinstein. [13] I ask the analogous question in this setting, and more generally, how they are glued together. To this end, I’d like to understand how the Borel–Smith conditions control gluing of local data.

Question 1.14. *Interpret the Borel–Smith conditions as cocycle conditions of descent data in $\mathrm{Spc}(\mathcal{K}(G))$.*

To do so, I will compute the local categories associated to the closed points $\mathcal{M}(H) \in \mathrm{Spc}(\mathcal{K}(G))$; other than the local category at $\mathcal{M}(1)$ which is simply the derived category, not much is known. Of particular interest is the point $\mathcal{M}(G)$ if G is a p -group, as the associated local category admits a description surprisingly similar to the setting of equivariant genuine spectra [11] via geometric fixed points.

Question 1.15. *Understand the local tt -categories associated to each closed point $\mathcal{M}(H) \in \mathrm{Spc}(\mathcal{K}(G))$, and in particular, compute the $\mathcal{M}(H)$ -local Picard groups of $\mathcal{K}(G)$.*

From these computations, one may use descent-theoretic arguments to give a possible answer to the first prominent question. Enhancing to the level of ‘2-rings’ and using the recent extension of tt -geometry to this setting by Aoki et al. [2] will be of use here, as descent in stable ∞ -categories is considerably more structured.

- Endotrivial modules, the invertible objects of $\mathrm{stmod}(kG)$, are famously tricky to describe in full generality, and their classification remains an interesting question. For p -groups, the classification was completed by Carlson–Thevenaz. [29, 30] For arbitrary groups, the question is still open, but considerable advances have been made in understanding the classification. See [65] for a comprehensive overview.

Recently, Balmer–Gallauer introduced the *stable permutation category* $\mathrm{stperm}(G; k)$, a permutation-enhanced analogue of the usual stable module category $\mathrm{stmod}(kG)$. [10] Methods from beyond

representation theory have been leveraged to understand endotrivial modules to great success; [4, 46, 60] we expect that one can use $\text{stperm}(G; k)$ to continue this trend.

Question 1.16. *Deduce the structure of $\text{Pic}(\text{stperm}(G; k))$ and use this to gain new understanding of $\text{Pic}(\text{stmod}(kG))$.*

1.6. Odds and ends. We close this section with a few related projects and a conjecture that do not fit in the above narrative.

- The notion of an orientable endotrivial complex can be viewed as a (field-dependent) extension of orientability of homotopy representations, as defined by tom Dieck–Petrie. [87] Since orientable representation spheres produce orientable endotrivials, it is pertinent to be able to determine when a representation sphere is orientable; yet this very natural question seems to never have been studied.

It is not hard to show that if an irreducible real representation is non-orientable, it must be of real type and nontrivial; one may ask if the converse holds.

Theorem 1.17. [74] *Let G be a finite group with normal Sylow 2-subgroup. Then an irreducible real representation V satisfies S^V is non-orientable if and only if it is of real type and nontrivial.*

This does not hold in general; for instance, J_2 and J_3 each have one real irreducible representation V for which S^V is orientable. The dimension function of a real representation (equivalently the h-mark function of the induced endotrivial) is a Borel-Smith function, and via the tom Dieck homomorphism (equivalently, its mod 2 reduction) can be realized as a unit in the Burnside ring (see e.g. [88]). We show that for 2-groups, the images of the irreducible real representations in the Burnside ring recover a basis of the unit group given by Bouc. [24]

The mod 2 reduction detects non-orientability at solvable subgroups. Quite surprisingly, this seems to completely characterize non-orientability.

Conjecture 1.18. *Let V be a real representation. Then S^V is orientable if and only if the dimension function $\dim_V : \text{Sub}(G) \rightarrow \mathbb{N}$ is constant modulo 2 on all solvable subgroups of G .*

We have verified this conjecture computationally for irreducible representations of all groups up to order 2000 and all finite simple groups in the GAP library `TomLib`.

- The *trivial source ring* $T(kG)$ is the triangulated Grothendieck ring of $\mathcal{K}(G)$, and was described in detail by Boltje–Carman. [22] The image of an endotrivial complex in $T(kG)$ is easily seen to be an orthogonal unit, and taking the image of an endotrivial complex defines the *Lefschetz homomorphism*. One may ask if this homomorphism is surjective; it is if G is a p -group [68] or if G has a cyclic Sylow p -subgroup [66] but in general, if k has 3 or more elements, we showed in joint work with Mazza there always exist groups for which the homomorphism is not. [66, 67] However, we showed in [73] that if $k = \mathbb{F}_2$, then the Lefschetz homomorphism is surjective, thus in a sense closing the question. This is established by translating the question into a question about morphisms of biset functors (‘global’ Mackey functors) and fusion-stability. The question is related to understanding the differences in formulations of Broué’s abelian defect group conjecture, in particular, comparing splendid Rickard equivalence and p -permutation equivalence.
- In [75], I gave a description of all *endosplit p -permutation resolutions*, finite p -permutation resolutions of a kG -module which are split upon tensoring with their dual. These can be considered a type of *relatively endotrivial complex* in the sense of [71], and play a role in proving Broué’s abelian defect group conjecture [52] in certain cases. I verified a Galois descent statement for these resolutions, and as an application, verified Kessar–Linckelmann’s Galois refinement of the abelian defect group conjecture [57] for certain blocks of p -solvable groups by adapting Harris–Linckelmann’s proof. The question of extending this step to all blocks of p -solvable groups remains open.

2. NONCOMMUTATIVE TENSOR-TRIANGULAR GEOMETRY AND FRAME THEORY

Nakano–Vashaw–Yakimov have developed a noncommutative analogue of tensor-triangular geometry [76, 77, 78, 79], i.e., for monoidal triangulated categories which need not have symmetric tensor product. This enables one to consider geometric questions for representation categories of non-cocommutative Hopf algebras or bimodules, for instance, and Nakano–Vashaw–Yakimov compute the Balmer spectra of certain examples.

Unsurprisingly, the theory is much more difficult, with straightforward statements in the tt setting, such as functoriality and classification of thick tensor ideals unclear. In fact, in [69] I gave an example of a functor between monoidal triangulated (mt) categories which does not induce a continuous map on noncommutative Balmer spectra.

In [69], I determined some situations where a mt-functor induces a continuous map on noncommutative Balmer spectra, and considered universality properties of different spectra in noncommutative tt-geometry. Nakano–Vashaw–Yakimov demonstrated that the noncommutative Balmer spectrum is the universal noncommutative support theory. I showed an analogous statement for the so-called complete prime spectrum for multiplicative support data, support data satisfying

$$\mathrm{supp}(x \otimes y) = \mathrm{supp}(x) \cap \mathrm{supp}(y)$$

for objects $x, y \in \mathcal{K}$ (noting for tt-geometry, the supports of $\mathrm{Spc}(\mathcal{K})$ always satisfy this). The question of understanding when support data is multiplicative is of interest to tensor categorists and Hopf algebraists. [20, 21]

Theorem 2.1. [69] *The complete prime spectrum $\mathrm{Spc}^{\mathrm{cp}}(\mathcal{K})$ of a mt-category $\mathcal{K}$ is the universal support amongst support data satisfying the tensor product property. Moreover, $\mathrm{Spc}^{\mathrm{cp}}(-)$ is functorial.*

One interesting question is to deduce when every prime in the noncommutative Balmer spectrum is completely prime; this would imply that the noncommutative Balmer spectrum satisfies good functoriality properties. Nakano–Vashaw–Yakimov gave one criterion for when this holds [77], and I gave a counterexample showing the converse does not hold in general [72]. The counterexample arises from diagrammatic representation theory, by showing existence of a semisimple subcategory of Khovanov’s Heisenberg category categorifying the Weyl algebra. [58]

Let $\mathcal{K}$ be an essentially small tt-category. One of the fundamental results of tensor-triangular geometry is ‘Balmer classification.’ Thomason subsets of $\mathrm{Spc}(\mathcal{K})$ are in order-preserving bijection with the (radical) thick tensor-ideals of $\mathcal{K}$. However, Huang–Vashaw [56] gave an example of a mt-category under which the bijection does not hold; instead, the thick tensor-ideals are parametrized by the specialization closed subsets, a strictly larger collection. De Deyn and I investigated this [34] and showed an analogous classification theorem holds for mt-categories, using frame theory and pointless topology to circumvent the bad topological properties the noncommutative Balmer spectrum can have.

Theorem 2.2. [34] *Let $\mathcal{K}$ be an essentially small mt-category. One has an order preserving bijection between the semiprime thick tensor ideals of $\mathcal{K}$ and the open subsets of the ‘pseudo-Hochster dual’ of the noncommutative Balmer spectrum.*

The key difference here, unlike in the symmetric setting, is that $\mathrm{Spc}(\mathcal{K})$ need not be a spectral space, so the Hochster dual need not be a proper ‘dual’ space; this is the mechanism that makes the usual classification theorem for tt-geometry fail. De Deyn and I also find conditions for when $\mathrm{Spc}(\mathcal{K})$ is spectral and give criteria for when Balmer classification holds on the nose. We’d like to understand conditions which force this: one nontrivial condition is for instance, $\mathrm{Spc}(\mathcal{K})$ being a Noetherian topological space. [83]

Question 2.3. *Determine intrinsic conditions on a monoidal triangulated category which force ‘Balmer classification’ to hold.*

The frame-theoretic view of tensor-triangular geometry had been considered before [26, 59] and recently has gained new interest, [45, 89] in part from the role of categorified locales in condensed mathematics. [1] Recently, Krause introduced the notion of a central subcategory and central support of a triangulated category [61] which satisfies good frame-theoretic properties; we’d like to understand the relation of this support theory to other support theories with good frame-theoretic behavior. Stevenson and I attempted to deduce central support for triangulated categories arising from a block algebra, $\mathrm{stmod}(kGb)$ or $D_b(\mathrm{mod}(kGb))$, and even here, it is quite an involved computation.

Question 2.4. *Relate Krause’s notion of a central subcategory and central support to other support theories for (monoidal-)triangulated categories, and compute examples.*

2.1. Bimodule categories. One natural mt-category which is not yet well-understood is the stable module category of left-right-projective (kGb, kGb) -bimodules, for b a block of kG . We denote this category by $\mathrm{lrp}(kGb)$. Solberg–Vashaw–Witherspoon [90] took the first steps in this direction and determined that if G is

a cyclic p -group, then $\mathrm{Spc}(\mathrm{lrp}(kG))$ is a point. In ongoing joint work with Vashaw, we compute $\mathrm{Spc}(\mathrm{lrp}(kG))$ when G is a p -group. Contrary to the tt-setting, the spectrum is not given by the cohomological support variety, the Hochschild cohomological support. The next step is to deduce the result for a block algebra.

Question 2.5. *Let b be a block of kG . Deduce the Balmer spectrum of $\mathrm{lrp}(kGb)$.*

We will begin by looking at principal blocks with trivial nucleus in the sense of [17], as one still has a grasp of all the thick subcategories. From here, one can consider the category of left-right projective p -permutation (kGb, kGb) -bimodules and the associated mt-category $K_b(\mathrm{lrpperm}(kGb)^\natural)$, the two-sided analogue of the usual derived category of permutation modules.

Question 2.6. *Let b be a block of kG . Deduce the Balmer spectrum of $K_b(\mathrm{lrpperm}(kGb)^\natural)$.*

This category is a key figure in Rickard's formulation of Broué's abelian defect group conjecture. [82] One implication of its statement is as follows: given two Brauer correspondent blocks kGb and $kN_G(D)c$, where D is an abelian defect group of b , one has an equivalence of monoidal triangulated categories

$$K_b(\mathrm{lrpperm}(kGb)^\natural) \cong K_b(\mathrm{lrpperm}(kN_G(D)c)^\natural).$$

We do not expect the noncommutative Balmer spectrum to be related to Hochschild cohomological support of the block, so this computation should provide a new invariant to either provide evidence for, or refute, the conjecture.

3. GEOMETRY OF OLIGOMORPHIC GROUPS AND BEYOND

Harman–Snowden [49] have introduced interesting new classes of tensor categories arising from oligomorphic groups, which are usually infinite but ‘combinatorially finite.’ These groups arise naturally from model theory as automorphism groups of Fraïssé limits; see [27, 49] for details.

In a direct analogue to the work I previously discussed, oligomorphic groups have a good notion of a G -set. Then, one can define a measure μ assigning ‘size’ of G -sets, and this enables one to construct a permutation module category $\mathrm{Perm}_k(G; \mu)$ over k . One also obtains a representation category $\mathrm{Rep}_k(G; \mu)$, and if μ is well-behaved, then $\mathrm{Rep}_k(G; \mu)$ is a pre-Tannakian tensor category and the abelian envelope of $\mathrm{Perm}_k(G; \mu)$. Even in situations where the measure μ is not well-behaved, one still can obtain a good representation category using different techniques. [32]

For instance, for $G = S_\infty$, choices of measure μ yield Deligne's interpolation categories $\mathrm{Rep}(S_t)$ and $\mathrm{Rep}^{\mathrm{ab}}(S_t)$. However, other examples, such as $G = \mathrm{Aut}(\mathbb{R}, <)$, the group of order-preserving automorphisms of the real line, provide the first known examples of pre-Tannakian categories that are not interpolation categories. [50] See also [32, 37, 48, 51].

In known examples, one has a finite permutation resolution result analogous to [7]. Harman and I are looking into this phenomena, and expect that under suitable hypotheses on the oligomorphic group G the measure μ , finite (p -)permutation resolutions should exist.

Question 3.1. *Determine conditions on G, μ for which every finite ‘ G -module’ $M \in \mathrm{Rep}_k(G; \mu)$ has a finite resolution by ‘ p -permutation G -modules’ in $\mathrm{Perm}_k(G; \mu)^\natural$. Construct explicit resolutions for simple modules.*

In some instances [32, 38], one finds quite a remarkable property: the category of G -representations $\mathrm{Rep}_k(G; \mu)$ admits a highest weight structure, with the p -permutation modules precisely the tilting modules of the category. Therefore, one obtains a tt-equivalence $D_b(\mathrm{Rep}_k(G; \mu)) \cong K_b(\mathrm{Perm}_k(G; \mu)^\natural)$, in stark contrast to the setting of finite groups. I would like to understand this phenomena better, as it does not always occur.

Question 3.2. *Determine criteria on G, μ that guarantee a tensor-triangulated equivalence $D_b(\mathrm{Rep}_k(G; \mu)) \cong K_b(\mathrm{Perm}_k(G; \mu)^\natural)$.*

In forthcoming work, Harman and I have deduced the tt-geometry of the representation categories associated to the groups $\mathrm{Aut}(\mathbb{R}, <)$ and $\mathrm{Aut}(\mathbb{S}^1, <)$, order-preserving automorphisms of the real line and 1-sphere, respectively. These categories are so-called *Delannoy categories*. In the case of $\mathrm{Aut}(\mathbb{R}, <)$, there are exactly 4 measures: [50] considers the first measure, which results in a semisimple category, and [32] considers the second (equivalently, third) measure.

Theorem 3.3. [47] *Let $\mathcal{A}$ denote either the abelian 2nd Delannoy category of [32] or the (finite) abelian circular Delannoy category of [51]. The Balmer spectrum $\mathrm{Spc}(\mathrm{D}_b(\mathcal{A}))$ is a two (resp. four) point space. Moreover, if $\mathcal{A}$ is the 2nd Delannoy category, then the rigidly-compactly generated tt-category $\mathrm{KInj}(\mathrm{Ind}\mathcal{A})$ is not stratified by $\mathrm{Spc}(\mathrm{D}_b(\mathcal{A}))$ in the sense of [12].*

In this paper, we also lay out some generalities for studying the tt-geometry of tensor categories. Non-stratification is generally rare for derived categories of tensor categories. Additionally, in both cases, the Balmer spectrum differs from the cohomological support of the category, which is somewhat uncommon. We’d like to compute more examples to discover more unusual behavior.

Question 3.4. *Deduce the tt-geometry for representation and permutation categories of oligomorphic groups.*

Nekrasov and I have begun studying the categories associated to the oligomorphic group $S_\infty \rtimes S_\infty$, which we expect should behave quite differently than the Delannoy categories.

Finally, the *higher Verlinde categories* Ver_{p^n} have emerged as critically important in the theory of tensor categories [14, 15]. These are examples of *incompressible* tensor categories [31], and the union of the categories Ver_p^∞ conjecturally receives fiber functors from all symmetric tensor categories over k of moderate growth.

Question 3.5. *Determine $\mathrm{Spc}(\mathrm{D}_b(\mathrm{Ver}_{p^n}))$ for primes p and positive integers n .*

This question is answered for Ver_8^+ (the only higher Verlinde category of tame representation type) in a forthcoming paper from Aguilar et al., but beyond this setting the question is wide open. The structure of the cohomology ring of Ver_{p^n} is conjectured but not known for sufficiently large n . [18] As the higher Verlinde categories are finite tensor categories, the Nakano–Vashaw–Yakimov conjecture predicts that $\mathrm{Spc}(\mathrm{D}_b(\mathrm{Ver}_{p^n}))$ is given by the homogeneous spectrum of its cohomology ring. [79]

APPENDIX A. GENERATIVE AI AND MATHEMATICS

Though this appendix may not be required, I believe that given the moment of time the global mathematics community is currently experiencing, I should state my views on generative AI in mathematics and on how generative AI use has contributed to my work. I will not discuss my ethical views on generative AI or my views on AI companies, as this is not the appropriate venue.

Recently, generative AI has reached a terrifyingly impressive ability to proof difficult theorems, shaking the field of mathematics. This development has caused a wide range of reactions, including excitement about what we will be able to answer in the future, questioning what our role in mathematics is, fear for the future of our profession and the world, and beyond. However, I remain casually optimistic. **I view mathematics as an artistic and philosophical pursuit as much as a theorem-proving business.** A beautiful, elegant idea or construction is recognizable to experienced mathematicians, but this may not manifest in ways that generative AI can presently ‘see’ as clearly, as this is a quality which is not so easily metricized (itself being subjective). Moreover, also important in mathematics are inquiry, understanding of results and context, developing new theories and perspectives, and engagement with the broader community.

Our work will manifest in different ways in the future; interpretation of generative AI output has been floated as a likely occurrence; bringing clarity to the output and forming new questions and theories from it will be of paramount importance. It is more important than ever that human mathematicians have understanding of the mathematics being developed; for me, any work I publish will always be completely human-written, and I must understand and be able to replicate every argument. Optimistically, productive uses of AI can allow us to actually spend more time on theory-building, asking philosophical questions, and seeking mathematical beauty and insight, while we delegate machines to do the tasks they excel at. In short, mathematicians may have the time and energy to **dream bigger**. However, I wholly agree with the statements many mathematicians have made about the threats that generative AI, if used poorly, will pose to the field of mathematics. These issues have been made clear and can be found in many a blog post or official statement, so I will not repeat them here.

AI usage in my work. Most of my prior projects have had little to no generative AI use. The exceptions are two papers [73, 74] which arose from experiments in August 2026 to see what generative AI was capable of at the time. In [74], I posed a generative AI, Claude Fable, a conjecture I had to see if it could make any progress. It recovered some results I had obtained, and proved a few new results I had not considered.

However, after enough poking and prodding, it found a counterexample to the conjecture. I eventually had Fable write GAP code to check for more examples, which ended up being highly productive. Interestingly, the paper contains a new conjecture which I, nor any generative AI (at present), do not know how to solve.

In [73], I posed Fable another lingering question from another project, [66]. It recognized that characteristic idempotents of fusion systems might be a viable target, and attempted (unsuccessfully) to prove it. However, I realized after a bit of literature review that the question had a fairly easy answer by using the framework of biset functors. Moreover, my advisor, Robert Boltje, had suggested looking at characteristic idempotents years prior, and this experiment prompted me to revisit those methods. Ultimately, I wrote up the paper with no generative AI assistance or proof writing, but I cannot deny that Fable played a role in my attaining the final result.

Beyond these papers, I have found in the past few months that generative AI has become better at scanning for errors, both mathematical and grammatical. While under review, ChatGPT 6 Pro found an subtly incorrect lemma in [70] that the anonymous reviewers did not find. This required me to find a new proof mechanism for a few major theorems in the paper. If generative AI keeps mathematicians' work in check, this is strictly a positive for mathematics.

REFERENCES

- [1] Ko Aoki. The sheaves–spectrum adjunction. *Math. Ann.*, 396(2):Paper No. 30, 2026.
- [2] Ko Aoki, Tobias Barthel, Anish Chedalavada, Tomer Schlank, and Greg Stevenson. Higher Zariski geometry. [arXiv:2508.11621](#), 2025.
- [3] Paul Balmer. Tensor triangular geometry. In *Proceedings of the International Congress of Mathematicians. Volume II*, pages 85–112. Hindustan Book Agency, New Delhi, 2010.
- [4] Paul Balmer. Stacks of group representations. *J. Eur. Math. Soc. (JEMS)*, 17(1):189–228, 2015.
- [5] Paul Balmer. Nilpotence theorems via homological residue fields. *Tunis. J. Math.*, 2(2):359–378, 2020.
- [6] Paul Balmer and Martin Gallauer. Permutation modules and cohomological singularity. *Comment. Math. Helv.*, 97(3):413–430, 2022.
- [7] Paul Balmer and Martin Gallauer. Finite permutation resolutions. *Duke Math. J.*, 172(2):201–229, 2023.
- [8] Paul Balmer and Martin Gallauer. Permutation modules, Mackey functors, and Artin motives. In *Representations of algebras and related structures*, EMS Ser. Congr. Rep., pages 37–75. EMS Press, Berlin, [2023] ©2023.
- [9] Paul Balmer and Martin Gallauer. The geometry of permutation modules. *Invent. Math.*, 241(3):841–928, 2025.
- [10] Paul Balmer and Martin Gallauer. Permutation, stabilization and decomposition. [arXiv:2604.17483](#), 2026.
- [11] Paul Balmer and Beren Sanders. The spectrum of the equivariant stable homotopy category of a finite group. *Invent. Math.*, 208(1):283–326, 2017.
- [12] Tobias Barthel, Drew Heard, and Beren Sanders. Stratification in tensor triangular geometry with applications to spectral Mackey functors. *Camb. J. Math.*, 11(4):829–915, 2023.
- [13] Tobias Barthel, Tomer M. Schlank, Nathaniel Stapleton, and Jared Weinstein. On Hopkins' Picard group. *Acta Mathematica*, 2026.
- [14] Dave Benson and Pavel Etingof. Symmetric tensor categories in characteristic 2. *Adv. Math.*, 351:967–999, 2019.
- [15] Dave Benson, Pavel Etingof, and Victor Ostrik. New incompressible symmetric tensor categories in positive characteristic. *Duke Math. J.*, 172(1):105–200, 2023.
- [16] Dave Benson, Srikanth B. Iyengar, and Henning Krause. Local cohomology and support for triangulated categories. *Ann. Sci. Éc. Norm. Supér. (4)*, 41(4):573–619, 2008.
- [17] Dave Benson, Srikanth B. Iyengar, and Henning Krause. Localising subcategories for cochains on the classifying space of a finite group. *C. R. Math. Acad. Sci. Paris*, 349(17–18):953–956, 2011.
- [18] David Benson and Pavel Etingof. On cohomology in symmetric tensor categories in prime characteristic. *Homology Homotopy Appl.*, 24(2):163–193, 2022.
- [19] David J. Benson, Jon F. Carlson, and Jeremy Rickard. Thick subcategories of the stable module category. *Fund. Math.*, 153(1):59–80, 1997.
- [20] Petter Andreas Bergh, Julia Yael Plavnik, and Sarah Witherspoon. Support varieties for finite tensor categories: the tensor product property. *Ann. Represent. Theory*, 1(4):539–566, 2024.
- [21] Petter Andreas Bergh, Julia Yael Plavnik, and Sarah Witherspoon. Support varieties without the tensor product property. *Bull. Lond. Math. Soc.*, 56(6):2150–2161, 2024.
- [22] Robert Boltje and Rob Carman. The orthogonal unit group of the trivial source ring. *J. Algebra*, 614(1):806–824, 2023.
- [23] Serge Bouc. The Dade group of a p -group. *Invent. Math.*, 164(1):189–231, 2006.
- [24] Serge Bouc. The functor of units of Burnside rings for p -groups. *Comment. Math. Helv.*, 82(3):583–615, 2007.
- [25] Glen E. Bredon. Equivariant cohomology theories. *Bull. Amer. Math. Soc.*, 73:266–268, 1967.
- [26] Aslak Bakke Buan, Henning Krause, and Øyvind Solberg. Support varieties: an ideal approach. *Homology Homotopy Appl.*, 9(1):45–74, 2007.
- [27] Peter J. Cameron. *Oligomorphic permutation groups*, volume 152 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1990.
- [28] Jon F. Carlson. The varieties and the cohomology ring of a module. *J. Algebra*, 85(1):104–143, 1983.
- [29] Jon F. Carlson and Jacques Thévenaz. The classification of endo-trivial modules. *Invent. Math.*, 158(2):389–411, 2004.

- [30] Jon F. Carlson and Jacques Thévenaz. The classification of torsion endo-trivial modules. *Ann. of Math. (2)*, 162(2):823–883, 2005.
- [31] Kevin Coulembier, Pavel Etingof, and Victor Ostrik. Incompressible tensor categories. *Adv. Math.*, 457:Paper No. 109935, 65, 2024.
- [32] Kevin Coulembier and Andrew Snowden. The second Delannoy category. [arXiv:2601.15574](#), 2026.
- [33] Ethan S. Devinatz, Michael J. Hopkins, and Jeffrey H. Smith. Nilpotence and stable homotopy theory. I. *Ann. of Math. (2)*, 128(2):207–241, 1988.
- [34] Timothy De Deyn and Sam K. Miller. Re-framing the classification of ideals in noncommutative tensor-triangular geometry. [arXiv:2510.23767](#), 2025.
- [35] Ronald M. Dotzel and Gary C. Hamrick. p -group actions on homology spheres. *Invent. Math.*, 62(3):437–442, 1981.
- [36] Ju.Á. Drozd. Tame and wild matrix problems. In *Representation theory, II (Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979)*, volume 832 of *Lecture Notes in Math.*, pages 242–258. Springer, Berlin, 1980.
- [37] Pavel Etingof and Andrew Snowden. Classification of simple commutative algebras in the Delannoy category. *Math. Z.*, 313(3):Paper No. 61, 25, 2026.
- [38] Johannes Flake and Jonathan Gruber. Monoidal Ringel duality and monoidal highest weight envelopes. [arXiv:2512.19558](#), 2025.
- [39] Eric M. Friedlander and Julia Pevtsova. Π -supports for modules for finite group schemes. *Duke Math. J.*, 139(2):317–368, 2007.
- [40] Yorick Fuhrmann. Modular fixed points in equivariant homotopy theory. [arXiv:2506.21413](#), 2025.
- [41] Martin Gallauer. Periods in equivariant and motivic contexts. [arXiv:2511.14325](#), 2025. With appendices by Ivo Dell’Ambrogio and Sam K. Miller.
- [42] Nick Georgakopoulos. The $RO(C_4)$ integral homology of a point. [arXiv:1912.06758](#), 2019.
- [43] Juan Omar Gómez and Sam K. Miller. The classification of integral endotrivial complexes. [arXiv:2605.31128](#), 2026.
- [44] Juan Omar Gómez and Sam K. Miller. Permutation twisted cohomology, reduced. Forthcoming, 2026.
- [45] Sira Gratz and Greg Stevenson. Approximating triangulated categories by spaces. *Adv. Math.*, 425:Paper No. 109073, 44, 2023.
- [46] Jesper Grodal. Endotrivial modules for finite groups via homotopy theory. *J. Amer. Math. Soc.*, 36(1):177–250, 2023.
- [47] Nate Harman and Sam K. Miller. The tensor-triangular geometry of some Delannoy categories. Forthcoming, 2026.
- [48] Nate Harman, Ilia Nekrasov, and Andrew Snowden. Arboreal tensor categories. *Selecta Math. (N.S.)*, 31(3):Paper No. 58, 46, 2025.
- [49] Nate Harman and Andrew Snowden. Tensor categories and oligomorphic groups. *Invent. Math.*, 2026.
- [50] Nate Harman, Andrew Snowden, and Noah Snyder. The Delannoy category. *Duke Math. J.*, 173(16):3219–3291, 2024.
- [51] Nate Harman, Andrew Snowden, and Noah Snyder. The circular Delannoy category. *Transform. Groups*, 2026.
- [52] M. E. Harris and M. Linckelmann. Splendid derived equivalences for blocks of finite p -solvable groups. *J. London Math. Soc. (2)*, 62(1):85–96, 2000.
- [53] Markus Hausmann and Stefan Schwede. Representation-graded Bredon homology of elementary abelian 2-groups. *Algebr. Geom. Topol.*, 25(7):4321–4340, 2025.
- [54] Lars Hesselholt and Piotr Pstrągowski. Dirac geometry I: Commutative algebra. *Peking Math. J.*, 8(3):405–480, 2025.
- [55] Michael J. Hopkins. Global methods in homotopy theory. In *Homotopy theory (Durham, 1985)*, volume 117 of *London Math. Soc. Lecture Note Ser.*, pages 73–96. Cambridge Univ. Press, Cambridge, 1987.
- [56] Hongdi Huang and Kent B. Vashaw. Group actions on monoidal triangulated categories and Balmer spectra. *Doc. Math.*, 30(5):1055–1083, 2025.
- [57] Radha Kessar and Markus Linckelmann. Descent of equivalences and character bijections. In *Geometric and topological aspects of the representation theory of finite groups*, volume 242 of *Springer Proc. Math. Stat.*, pages 181–212. Springer, Cham, 2018.
- [58] Mikhail Khovanov. Heisenberg algebra and a graphical calculus. *Fund. Math.*, 225(1):169–210, 2014.
- [59] Joachim Kock and Wolfgang Pitsch. Hochster duality in derived categories and point-free reconstruction of schemes. *Trans. Amer. Math. Soc.*, 369(1):223–261, 2017.
- [60] Achim Krause. The Picard group in equivariant homotopy theory via stable module categories. *J. Topol.*, 18(2):Paper No. e70020, 41, 2025.
- [61] Henning Krause. Central support for triangulated categories. *Int. Math. Res. Not. IMRN*, (22):19773–19800, 2023.
- [62] Wolfgang Lück. *Isomorphism conjectures in K- and L-theory*, volume 78 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer, Cham, [2025] ©2025. With contributions by Christoph Winges.
- [63] Akhil Mathew, Niko Naumann, and Justin Noel. Nilpotence and descent in equivariant stable homotopy theory. *Adv. Math.*, 305:994–1084, 2017.
- [64] Akhil Mathew and Vesna Stojanoska. The Picard group of topological modular forms via descent theory. *Geom. Topol.*, 20(6):3133–3217, 2016.
- [65] Nadia Mazza. *Endotrivial modules*. SpringerBriefs in Mathematics. Springer, Cham, 2019.
- [66] Nadia Mazza and Sam K. Miller. The Euler characteristic of an endotrivial complex. *J. Pure Appl. Algebra*, 230(10):Paper No. 108345, 2026.
- [67] Sam K. Miller. Endotrivial complexes. *J. Algebra*, 650:173–218, 2024.
- [68] Sam K. Miller. The classification of endotrivial complexes. *Adv. Math.*, 478:Paper No. 110404, 40, 2025.
- [69] Sam K. Miller. On functoriality and the tensor product property in noncommutative tensor-triangular geometry. [arXiv:2505.01899](#), 2025.
- [70] Sam K. Miller. Permutation twisted cohomology, remixed. [arXiv:2509.00954](#), 2025.
- [71] Sam K. Miller. Relatively endotrivial complexes. *J. Pure Appl. Algebra*, 229(2):Paper No. 107867, 38, 2025.
- [72] Sam K. Miller. A semisimple subcategory of Khovanov’s Heisenberg category. [arXiv:2512.13968](#), 2025.
- [73] Sam K. Miller. The fusion-stable tom Dieck homomorphism. [arXiv:2608.12499](#), 2026.
- [74] Sam K. Miller. Non-orientable representation spheres. [arXiv:2608.18015](#), 2026.
- [75] Sam K. Miller. On endosplit p -permutation resolutions and Broué’s conjecture for p -solvable groups. *Represent. Theory*, 30:151–174, 2026.

- [76] Daniel K. Nakano, Kent B. Vashaw, and Milen T. Yakimov. Noncommutative tensor triangular geometry. *Amer. J. Math.*, 144(6):1681–1724, 2022.
- [77] Daniel K. Nakano, Kent B. Vashaw, and Milen T. Yakimov. Noncommutative tensor triangular geometry and the tensor product property for support maps. *Int. Math. Res. Not. IMRN*, (22):17766–17796, 2022.
- [78] Daniel K. Nakano, Kent B. Vashaw, and Milen T. Yakimov. On the spectrum and support theory of a finite tensor category. *Math. Ann.*, 390(1):205–254, 2024.
- [79] Daniel K. Nakano, Kent B. Vashaw, and Milen T. Yakimov. The homological spectrum for monoidal triangulated categories. [arXiv:2506.19946](#), 2025.
- [80] Amnon Neeman. The chromatic tower for $D(R)$. *Topology*, 31(3):519–532, 1992. With an appendix by Marcel Bökstedt.
- [81] Douglas C. Ravenel. Localization with respect to certain periodic homology theories. *Amer. J. Math.*, 106(2):351–414, 1984.
- [82] Jeremy Rickard. Splendid equivalences: derived categories and permutation modules. *Proc. London Math. Soc.* (3), 72(2):331–358, 1996.
- [83] James Rowe. Noncommutative tensor triangular geometry: classification via Noetherian spectra. *Pacific J. Math.*, 330(2):355–371, 2024.
- [84] Greg Stevenson. A tour of support theory for triangulated categories through tensor triangular geometry. In *Building bridges between algebra and topology*, Adv. Courses Math. CRM Barcelona, pages 63–101. Birkhäuser/Springer, Cham, 2018.
- [85] Richard W. Thomason. The classification of triangulated subcategories. *Compositio Math.*, 105(1):1–27, 1997.
- [86] Tammo tom Dieck. *Transformation groups*, volume 8 of *De Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1987.
- [87] Tammo tom Dieck and Ted Petrie. Homotopy representations of finite groups. *Inst. Hautes Études Sci. Publ. Math.*, (56):129–169, 1982.
- [88] Ergün Yalçın. An induction theorem for the unit groups of Burnside rings of 2-groups. *J. Algebra*, 289:105–127, 2005.
- [89] Changhan Zou and Jiacheng Liang. Smashing, Balmer, Zariski spectra: an ideal approach. [arXiv:2607.13329](#), 2026.
- [90] Øyvind Solberg, Kent B. Vashaw, and Sarah Witherspoon. Noncommutative tensor triangular geometry: modules, bimodules, and unipotent Hopf algebras. [arXiv:2511.10531](#), 2025.