Hops, Checkers, and Fibonaccis!

Combinatorial Games & Counting Arguments

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Question 1.

Why do we do mathematics?

The practice of mathematics cultivates virtues that help people flourish. And the movement towards virtue happens through basic human desires. I want to talk about five desires we all have. The first of these is play...
Mathematics makes the mind its playground. We play with patterns, and within the structure of certain axioms, we exercise freedom in exploring their consequences, joyful at any truths we find...
Mathematical play builds virtues that enable us to flourish in every area of our lives. For instance, math play builds hopefulness: when you sit with a puzzle long enough you are exercising hope that you will eventually solve it. Math play builds community—when you share in the delight of working on a problem with another human being...
Play is part of human flourishing. You cannot flourish without play.

- Francis Su, "Mathematics for Human Flourishing,"
Retiring MAA Presidential Address, 2017

(the other four desires are: **beauty, truth, justice,** and **love!**)

Question 2.

What really is mathematics?

Most mathematical activity involves the use of pure reason to discover or prove the properties of abstract objects, which consist of either abstractions from nature or — in modern mathematics — entities that are stipulated with certain properties.

- Wikipedia page for "Mathematics"

My goals today:

1. Paint a picture of the playfulness of mathematics.
2. Survey some of the types of questions that captivated me from a younger age.
3. Give a gentle taste of proofs and mathematical logic.
4. Make you flex those brain muscles a bit!

We won't discuss my current research today - that's a bit less gentle :)

We'll start with research I did in undergrad!

The Knight's Tour

In the game of Chess, a *Knight* moves in an "L" shape: 2 moves vertically and 1 move horizontally, or vice versa.

Goal: On an 8×8 chessboard, move the Knight repeatedly move the Knight so that it touches each square exactly once.

0	47	18	33	14	61	20	35
17	32	63	48	19	34	13	60
50	1	46	15	62	11	36	21
31	16	49	10	37	22	59	12
2	51	30	45	8	55	24	39
29	44	9	54	23	38	7	58
52	3	42	27	56	5	40	25
43	28	53	4	41	26	57	6

Varieties of the Knight's Tour:

- Open Knight's Tour
- Closed Knight's Tour
- *Challenging* Knight's Tour

Theorem (Schwenk, 1991)

The Closed Knight's Tour is possible on an $m \times n$ chessboard, with $m \leq n$, unless:

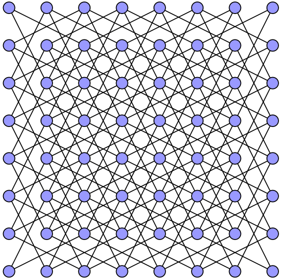
- m and n are both odd.
- $m = 1, 2,$ or 4.
- $m = 3$ and $n = 4, 6,$ or 8.

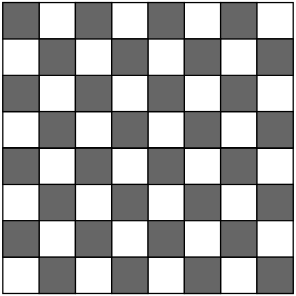
The *Challenging* Knight's Tour has been computationally verified in various cases via computer. We will prove it is possible *without computer assistance!*

We can represent all the possible knight moves with a *graph*, which consists of *vertices*, and *edges* connecting them.

The Knight's Tour Graph

The *complete knight's tour graph* has 64 vertices and 168 edges, with vertices x and y connected if a knight can move from square x and square y.

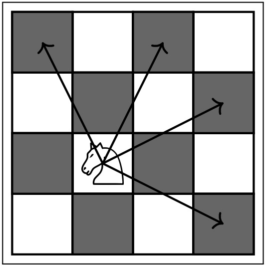




The Challenging Knight's Tour, Rephrased

Given a specified start and end vertex, does a *Hamiltonian Path* exist for the complete knight's tour graph?

Note that a knight changes color when it jumps! Therefore, two vertices have opposite colors if and only if they are odd distance apart. Therefore, a chessboard graph has no cycles of odd length.



Since we must make 63 moves to finish the tour, we must end on opposite colors. Therefore, if the start and end vertices have the same color, a Hamiltonian path cannot exist!

Definition

Call a chessboard graph *traversable* if we may find a Hamiltonian path from any two vertices of opposite color.

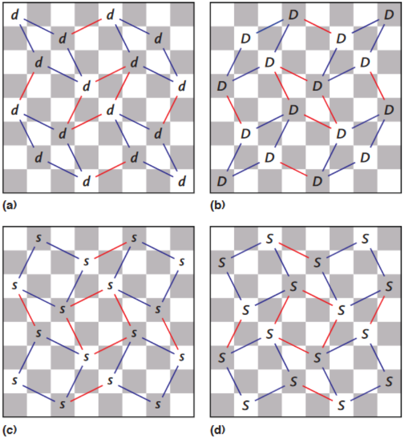
The Challenge

Is the 8×8 chessboard graph traversable?
If we select two vertices with *opposite color*, (i.e. odd distance apart) o and e, can we find a Hamiltonian path starting on o and ending on e?

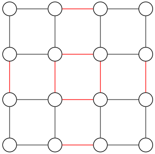
(Benjamin, M.)

The 8×8 chessboard graph is traversable!

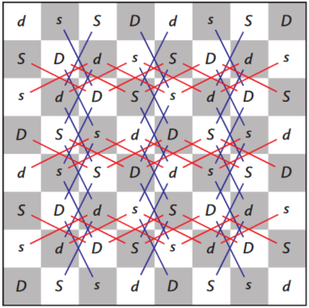
To begin, we will partition the chessboard graph into four subgraphs, a divide-and-conquer strategy.



(a) and (b) are *diamond*-type systems, and (c) and (d) are *square*-type systems.



All four systems are isomorphic to $G_{4,4}$.

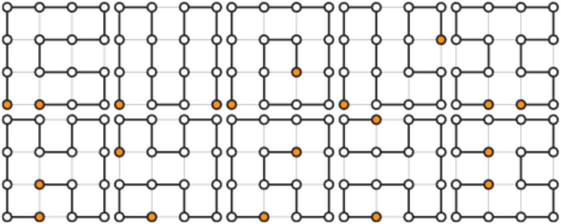


Other than the corner squares, it is always possible to hop between systems of different type. However, it is not possible to hop from one system to the other of same type.

Lemma

$G_{4,4}$ is traversable.

Proof. We given an explicit construction for each possible traversal, up to rotation and reflection.



□

So in a system, we can start and end anywhere, as long as the start and end are on opposite colors!

Our strategy to complete the Knight's Tour is to traverse each system, one at a time, then hop to the next. Completing all four systems means a complete knights tour!

Our plan will differ slightly depending on which systems the start o and end e belong to.

Three Cases

There are three cases to consider, of increasing difficulty.

- Opposite system types (ex: $o \in S$, $e \in d$)
- Same system types (ex: $o \in S$, $e \in s$)
- Same system (ex: $o, e \in S$)

We'll prove that all three cases may be traversed.

Case 1: Opposite System Types

Start at o and traverse the first system of type A, making sure to end in one of the middle four squares m_1.

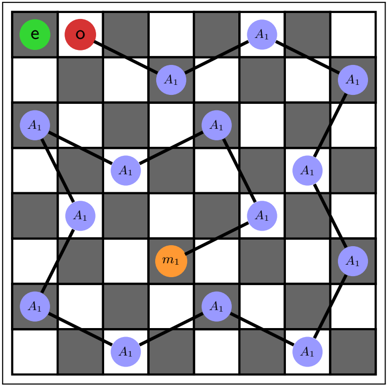


Figure: $o \xrightarrow{A_1} m_1$

Jump into the system of type B that does not contain e and traverse it, ending in a middle square m_2.

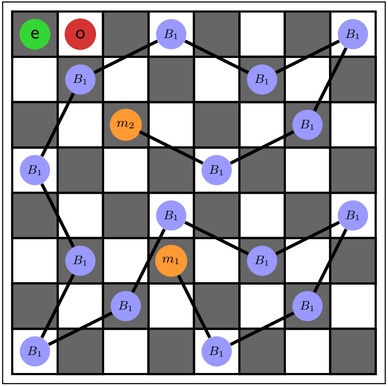


Figure: $m_1 \xrightarrow{B_1} m_2$

Jump into A_2 and traverse it, ending in a middle square m_3.

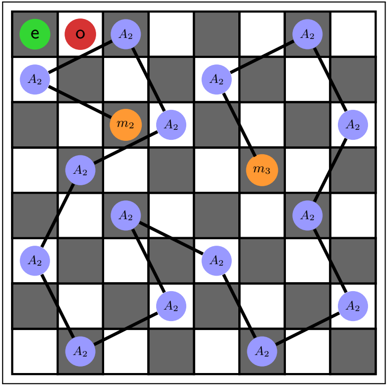


Figure: $m_2 \xrightarrow{A_2} m_3$

Jump into B_2 and traverse it, ending at e, and we're done!

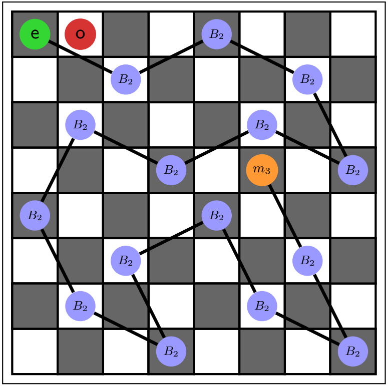


Figure: $m_3 \xrightarrow{B_2} e$

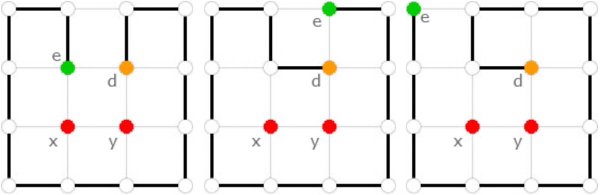
Case 2: Same System Type

Since we have to start and end in the same system type, we cannot simply proceed through all four systems. We'll need to switch things up a bit.

Lemma

Given any vertex e in $G_{4,4}$, we can find three middle vertices d, x, y, with x and y adjacent, such that a 13-step path from d to e reaches every vertex except x and y. Call this a *semitraversal*.

Proof. We compute explicitly:



□

Start at o and traverse the first system of type A, making sure to end in one of the middle four squares m_1. Note the locations of d, x, and y based on e.

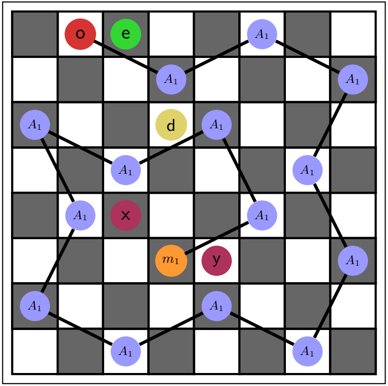


Figure: $o \xrightarrow{A_1} m_1$

Jump into a system of type B and traverse it, ending in a square adjacent to x or y (depending on which has the color of o), m_2, then jump through x and y.

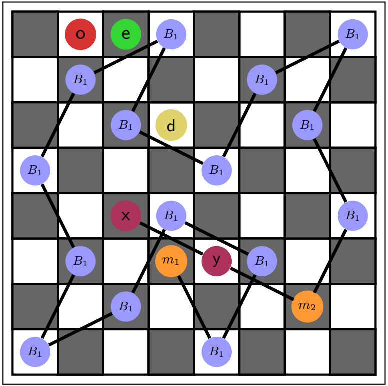


Figure: $m_1 \xrightarrow{B_1} m_2 \to y \to x$

Since x and y are in system A_2 and in the center of the board, we can jump to system B_2. Traverse system B_2, ending at a square next to d, m_3.

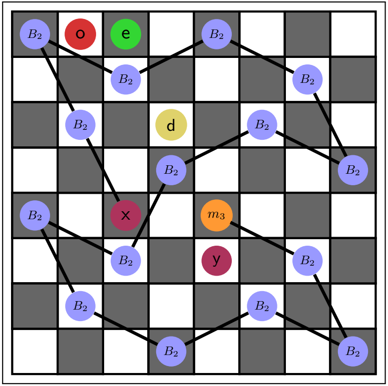


Figure: $x \xrightarrow{B_2} m_3$

Jump to d and complete the semitraverse of A_2 ending on e, and we're done!

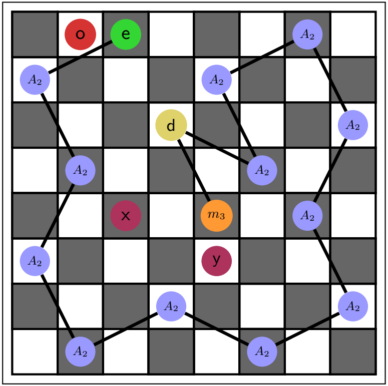


Figure: $m_3 \xrightarrow{A_2 - \{x,y\}} e$

Case 3: Same System

Move as if you were to traverse A_1 starting on o and ending on e, but stop on your penultimate move, at n_1. Find a vertex f in system B_2 adjacent to e.

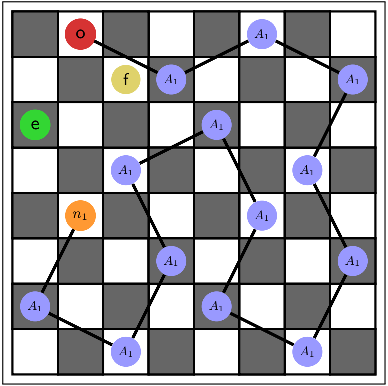


Figure: $o \xrightarrow{A_1 - \{e\}} n_1$

From n_1, traverse through B_1 and A_2 as usual.

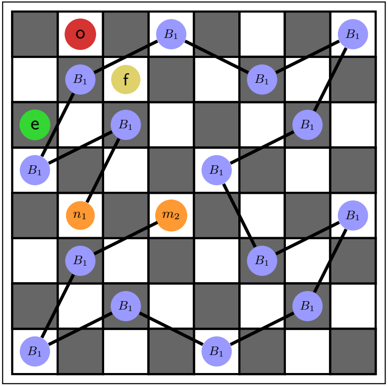


Figure: $n_1 \xrightarrow{B_1} m_2$

From n_1, traverse through B_1 and A_2 as usual.

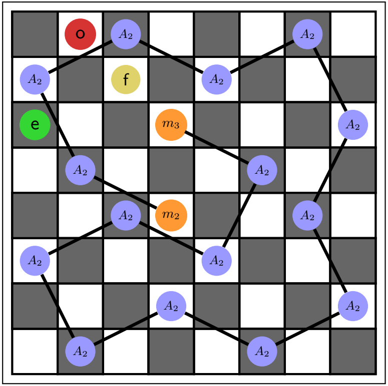


Figure: $m_2 \xrightarrow{A_2} m_3$

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Same System
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Traverse B_2 ending on f, then hop to e and we are done!

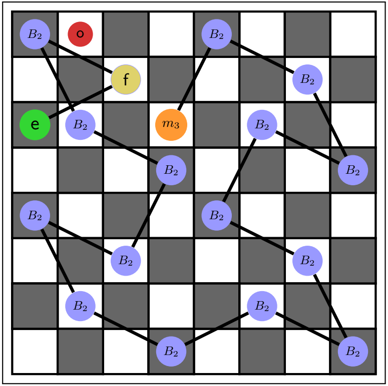


Figure: $m_3 \xrightarrow{B_2} f \rightarrow e$

Or are we? This strategy does not work when e or n_1 is in the corner of the board!

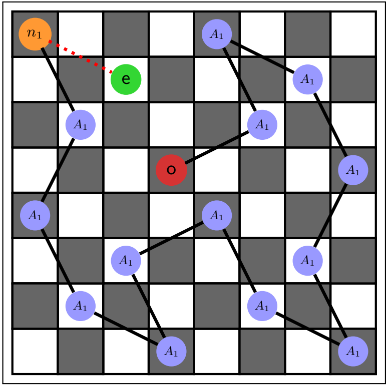


Figure: After reaching n_1, we are trapped.

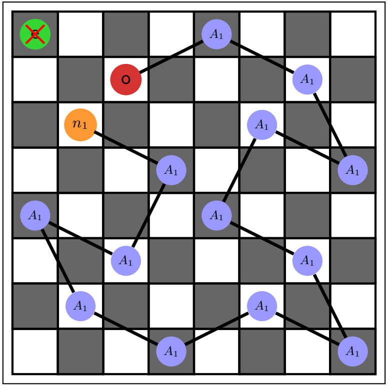


Figure: After leaving A_1 we cannot reach e.

Call n_0 and n the 3rd and 4th to last vertices in the traverse of A_1.

If n_1 is in the corner, leave two moves earlier at n, then after traversing B_1, A_2, and B_2, finish A_1 starting at n_0, and we're done!

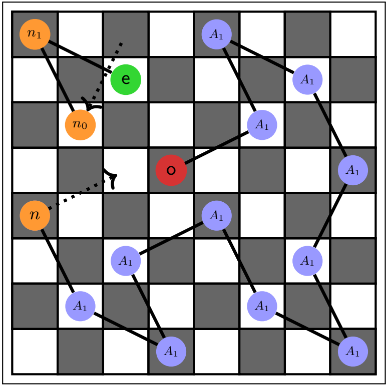


Figure: $o \xrightarrow{A_1 - \{n_0, n_1, e\}} n \xrightarrow{B_1, A_2, B_2} n_0 \to n_1 \to e$

If e is in the corner, leave one move earlier at n_0, then after traversing B_1, A_2, and B_2, finish A_1 starting at n_1, and we're done!

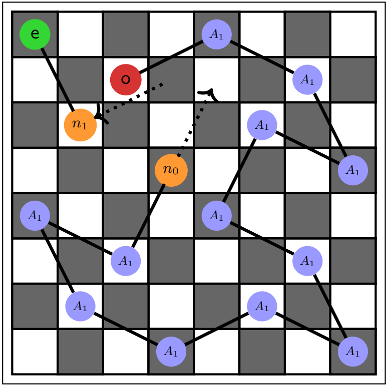


Figure: $o \xrightarrow{A_1 - \{n_1, e\}} n_0 \xrightarrow{B_1, A_2, B_2} n_1 \rightarrow e$

Since all cases have been demonstrated to be possible, we can conclude that **the 8×8 chessboard is traversable**, i.e. the Challenging Knight's Tour is possible, so long as the start and end squares are on opposite colors!

Corollary (M.)

The Challenging Knight's Tour can be solved for any board of size $2m \times 4n$ for $m \geq 4, n \geq 2$.

This may be proven by showing that any $G_{m,2n}$ is traversable and subtraversable (an inductive argument suffices), and generalizing our strategies for the three placement cases.

This paper was written with the help and advice of Arthur T. Benjamin, whose DVD lecture series "The Joy of Mathematics" was one of my first experiences with mathematical logic in middle school!

The next two topics were first introduced to me by him.

Let's introduce a new puzzle, first studied by John Conway in 1961, *Conway's Soldiers*.

The Rules

- We play on an infinite 2-dimensional grid, which is cut in half by a horizontal line. The top half is enemy territory and the bottom half is the friendly territory.
- First, you may place any (finite) amount of checkers in the friendly territory. After, you begin moving.
- Checkers move vertically or horizontally, are only allowed to hop over each other, and any pieces that is hopped over is removed from play.
- **The Goal:** Get a checker n rows into enemy territory.
- **The Question:** What is the minimum number of checkers needed to accomplish the goal for set values of n?

Let's try it out for small n! $n = 1$?

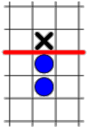


Figure: 2 checkers are necessary

$n = 2$?

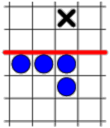


Figure: 4 checkers are necessary

$n = 3$?

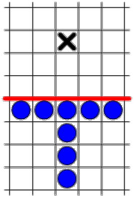


Figure: 8 checkers are necessary

$n = 4$?

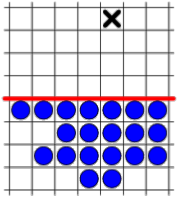


Figure: 20 checkers are necessary

$n = 5$?

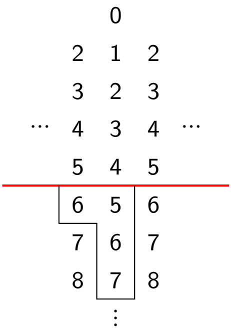
Theorem (Conway):

For any finite placement of checkers, it is impossible to move 5 rows into enemy territory.

Proof: Consider a square p on row 5, and label that square 0. We will show we cannot reach this square. For every other square, label that square by the number of steps it would take to move to p by moving vertically or horizontally.

		0		
	2	1	2	
	3	2	3	
$\cdots$	4	3	4	$\cdots$
	5	4	5	

For any placement of pieces, we give each checker in the placement weight g^k, where k is the label of the square, and $g = (\sqrt{5} - 1)/2$. We say the *weight of the configuration* is the sum of the weights of all the checkers.



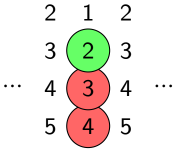
As an example, under this labelling, the row 2 configuration would have weight

$$(g^7 + g^6) + (g^6 + g^5).$$

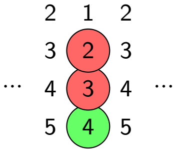
Now, g has the special property that $g^2 + g = 1$. By multiplying each side by g^n, it follows that $g^{n+2} + g^{n+1} = g^n$ for any integer n. So we can simplify this example to

$$(g^7 + g^6) + (g^6 + g^5) = (g^5 + g^4) = g^3.$$

If a checker jumps to a square with lesser labeling, we replace two checkers of weights g^k, g^{k-1} with one checker of weight g^{k-2}, and since $g^k + g^{k-1} = g^{k-2}$, the weight of the new configuration **is the same.**



If the checker jumps to a square with greater labeling, we replace two checkers of weight g^k, g^{k+1} with a checker of weight g^{k+2}, and since $g^k + g^{k+1} > g^{k+2}$ (as in general, $g^i > g^{i+1}$ for positive i), the weight of the resulting configuration **decreases.**



After we make any number of moves, the resulting placement's weight must be **less than or equal to the starting placement's weight**.

For any placement that ends on the square p in row 5, its initial weight must have been **at least 1**. Now, let's compute a bound on *maximal possible initial weight of any valid placement,* by supposing there is a checker placed on every square.

	6	5	6	
⋯	7	6	7	⋯
	8	7	8	
		⋮		

In row 1, the weight must be bounded by

$$(g^5 + g^6 + g^7 + \dots) + (g^6 + g^7 + \dots) = g^5/(1-g) + g^6/(1-g).$$

However, observe:

$$g^k + g^{k-1} = g^{k-2} \iff g^k = g^{k-2} - g^{k-1} = g^{k-2}(1-g) \iff g^k/(1-g) = g^{k-2}.$$

Therefore, we can simplify further:

$$g^5/(1-g) + g^6/(1-g) = g^3 + g^4 = g^2,$$

hence any placement's weight in the first row must be bounded by g^2.

	6	5	6	
⋯	7	6	7	⋯
	8	7	8	
		⋮		

Similarly, in row 2, the weight must be bounded by

$$(g^6+g^7+g^8+\dots)+(g^7+g^8+\dots)=g((g^5+g^6+g^7+\dots)+(g^6+g^7+\dots))=g^3.$$

The weight in the kth row is bounded by g^{k+1}. So what is the final upper bound on the weight?

$$g^2+g^3+g^4+\dots=g^2/(1-g)=g^0=1.$$

So *any* finite placement must have starting weight strictly less than 1! But since a starting weight of at least 1 is necessary to reach the square p, we cannot reach p! □

What if we can (somehow) place *infinitely* many checkers?

Theorem: (Tatham, Taylor)

If the player is allowed to place infinitely many checkers in their starting area, it is possible to reach row 5 after infinitely many moves.

What if we expand the game to higher dimensions?

Theorem: (Eriksson, Lindstrom)

In the n-dimensional variant of Conway's Checkers, it is impossible to reach the $(3n-1)$th row, and always possible to reach the $(3n-2)$th row.

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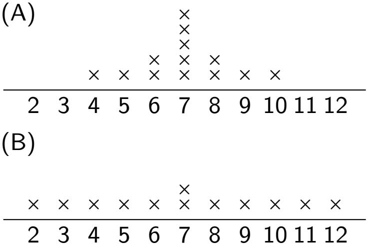
Let's look at an old betting game, Knock Em Down, and try to find the optimal play!

The Classical Game

- Each player is given 12 tokens, and puts them on their board, which consists of columns labeled 2 to 12.
- Two dice are rolled, the results are added, and each player removes a token from the corresponding column, if they have any left.
- Repeat until a player runs out of tokens. The first player to remove all their tokens wins!



Which player is more likely to win?

A beats B 75% of the time, B beats A 16% of the time, and there is a draw 9% of the time.

Note that the probability of rolling a 7 is $1/6$, while the probability of rolling a 2 is $1/36$, as is the probability of rolling a 12.

Let's look at a simpler variant.

	(A)	(B)
1		
2,3	$\times$	
4,5,6	$\times\times$ $\times\times$	$\times\times$ $\times\times\times$

Which player wins?

If a 2 or 3 is rolled at any point, (A) wins, and otherwise (B) wins. There cannot be a draw! (A) wins are roughly 92% of the time.

How about this scenario?

	(A)	(B)
1	×	×
2,3	××	×
4,5,6	××	×××

- If a 2/3 is rolled twice, then (A) has an advantage.
- On the other hand, if a 4/5/6 is rolled thrice, then (B) has an advantage.
- If a 1 is not rolled until both the 2/3 and 4/5/6 boxes are cleared out, a tie occurs.

These placements are very evenly matched, with (A) winning 35.3% of the time, and (B) winning 33.6% of the time. A draw happens nearly a third of the time!

In both scenarios, (A) was closer to the *probability histogram* of the game than (B).

	(A)	(B)
1		
2,3	×	
4,5,6	×× ××	×× ×××

	(A)	(B)
1	×	×
2,3	××	×
4,5,6	××	×××

Perhaps the optimal strategy is to follow the histogram as close as possible?

In this variant, there are 10 outcomes and 10 tokens.

	(A)	(B)
1	×	
2,3	××	××
4,5,6	×××	×××
7,8,9,10	×× ××	×× ×××

Our hypothesis suggests (A) wins more frequently.

However, the **expected value** (the weighted average) of how many rolls it takes for (B) to clear is lower. $E[X_B] = 16.3$, while $E[X_A] = 17.7$.
In fact, (B) has the lowest expected value among *all* placements, making it a global **minimal allocation**. So our hypothesis is incorrect...

Wait! We theorized that (A) beats (B) more frequently, not that (B) on average finishes faster than (A). These are not necessarily the same question!

	(A)	(B)
1	×	
2,3	××	××
4,5,6	×××	×××
7,8,9,10	×× ××	×× ×××

By the same logic as we've used, if a 1 is rolled before five 7/8/9/10s are rolled, (A) must win! This is in fact likely to happen!

Despite (B) on average taking less time to finish, (A) beats (B) more often, 36% to 23%, with a draw the most likely outcome, at 41%. In fact, (A) can be shown to be likely to beat *all possible placements*.

Definition

Call a placement on a variant of Knock Em Down which has a probabilistic advantage over any other placement an *emperor*.

So to "solve" Knock Em Down, it suffices to find an emperor.

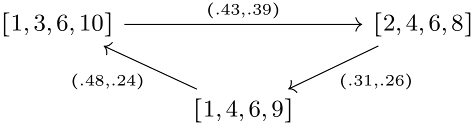
New question

When is there an emperor?

Let's take another look at the previous game board, which we'll denote $P(.1, .2, .3, .4)$ (each argument representing the probability of being chosen), but now with 20 tokens.

	(A)	(B)	(C)
1	1	2	1
2,3	3	4	4
4,5,6	6	6	6
7,8,9,10	10	8	9

Here, (B) follows the histogram exactly, while (A) and (C) are a bit off. On the other hand, it can be explicitly computed that (A) has the fastest expected time to finishing. How do they match up?

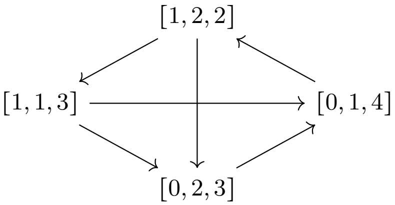


None of these are the emperor!

Definition

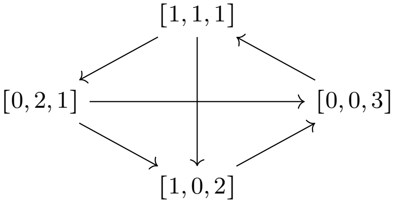
A variant of Knock Em Down with no emperor is a *non-transitive* game.

Going back to the game given by $P(1/6, 1/3, 1/2)$ with 5 tokens, we find it is also non-transitive, with the matchup of the five best placements as follows:



What is the *smallest* non-transitive game?

A three-token setup exists, with a caveat. If we restrict the $P(1/6, 1/3, 1/2)$ variant to 3 tokens, then $[0, 1, 2]$ is an emperor! However, if we ban that configuration, we reach a non-transitive situation:



It is unknown whether there are other non-transitive 3-token games. However, this is the "smallest" possible game:

Theorem (Benjamin)

A 2-token game must have an emperor.

What about large games?

Theorem (Nelson):

As the number of tokens grows, the probability of an emperor existing goes to 0.

In short, variants with a large number of tokens are unlikely to have an emperor. Why should this be?

Strategy: "Undercutting"

We illustrate with an example: consider the game given by $P(1/3, 1/3, 1/3)$ with 3000 tokens.

- If player (A) uses the allocation [1000,1000,1000], player (B) can "undercut" by using [999,999,1002]. If we play until (B) finishes, the odds column 3 finished last is around $.36$, and this is the only way (B) loses.
- Player (C) can undercut (B) by using [998, 1001, 1001], as the only way (C) loses is if column 2 finishes last in their game.
- However, player (A) is favored against (C), since (A) loses only when their last column is 1.

This strategy can be utilized to demonstrate lack of existence of an emperor.

In general, determining for an *arbitrary* variant, whether an emperor exists or what the minimal allocation is, is open (and difficult). However, some cases are known.

Theorem

If there are only two columns in a game variant, with probability distribution $P(p, 1-p)$ and t tokens, the minimal allocation is $[m, t-m]$, where m is the pth percentile of the distribution Bin(t, p).

In the case of $P(2/3, 1/3)$ (corresponding to dice rolls of 1,2 vs. 3,4,5,6) with 9 tokens,

- the emperor is [7,2],
- and the minimal allocation is [6,3]

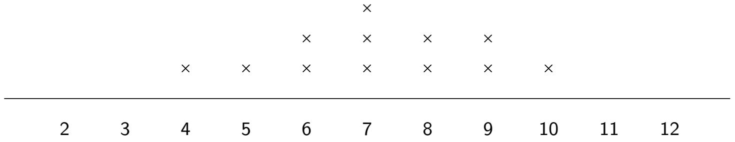
While an emperor may not exist, a minimal allocation must exist!

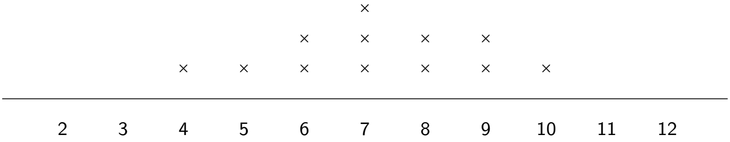
Theorem (Benjamin)

Let $x^* = (x_1, \ldots, x_n)$ be a minimal allocation for a t token game with $P(p_1, \ldots, p_n)$ probability vector. Then x^* satisfies:

1. If $p_i < p_j$, $x_i \leq x_j$.
2. If $p_i = p_j$, $|x_i - x_j| \leq 1$.
3. If $p_i < p_j$, then $(x_i - 1)/x_j < p_i/p_j$.

This drastically reduces the number of cases to check to determine an emperor. In the case of the original game, it reduces the number of cases from 646,646 to 49. One minimal allocation is:





This placement is believed to be an emperor, but a positive answer has not yet been proven. It is a *local emperor*, in that it is likely to beat any other configuration that differs by one token.

Some final open problems:

- When is a local emperor an emperor?
- If x is a minimal allocation for a variant with t tokens, then there is a minimal allocation x' for the same variant with $t+1$ tokens which contains x.
- The minimal allocation cannot differ too much from the probability histogram P. Precisely,

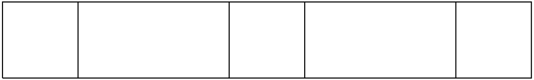
$$\lim_{t\to\infty} \frac{x^(t)}{t} = P.$$

In enumerative combinatorics, we count the number of "combinatorial objects" that exist, which have constraints indexed by the natural numbers.

If that's a bit confusing, let's give an example.

Question:

How many ways are there to tile a $1 \times n$ grid with 1×1 squares and 1×2 dominoes, where all squares and dominoes are indistinguishable?



Claim: The Fibonacci number F_n, defined recursively via:

$$F_0 := 1, F_1 = 1, F_n := F_{n-1} + F_{n-2},$$

counts the number of ways to tile a $1 \times n$ grid.

Claim: The Fibonacci number F_n, defined recursively via:

$$F_0 := 1, F_1 = 1, F_n := F_{n-1} + F_{n-2},$$

counts the number of ways to tile a $1 \times n$ grid.

Proof: Let T_n count the number of ways to tile a $1 \times n$ grid. Note that $F_1 = T_1$ and $F_0 = T_0$. Now, for any valid tiling of the $1 \times n$ grid, it can end in only two possible ways: with a square or with a domino.

A valid tiling that ends with a square is the same as a valid tiling of a $1 \times (n-1)$ grid, with a square tacked on at the end! Therefore, T_{n-1} counts the number of tilings of the $1 \times n$ grid that end with a square.

By a similar argument, T_{n-2} counts the number of tilings that end with a domino! Since all tilings must end with either a square or a domino, we conclude that $T_{n-2} + T_{n-1} = T_n$. Since the formulas for F_n and T_n match, they must be equal! □

Using this new interpretation of the Fibonaccis, let's prove some identities! We will prove these by "double counting," i.e. counting the same thing in two ways.

Theorem:

$F_0 + F_1 + \cdots + F_n = F_{n+2} - 1$

Proof: We ask the question, "How many $n+2$-tilings are there that use at least one domino?

- (A1): There is only one tiling that doesn't use a single domino, therefore there are $F_{n+2} - 1$ such tilings.
- (A2): Consider the location of the rightmost domino. If the domino ends at the $(k+2)$th tile, then the k tiles before give a $1 \times k$-tiling, of which there are F_k total, but all tiles after must be covered by squares. Summing over all possibilities, this gives $F_0 + F_1 + F_2 + \cdots + F_n$ total tilings.

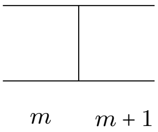
□

Theorem:

$F_{m+n} = F_m F_n + F_{m-1} F_{n-1}$.

Proof: How many $m+n$-tilings are there?

- (A1) F_{m+n}, by definition.
- (A2) Consider the line at the end of the mth grid:



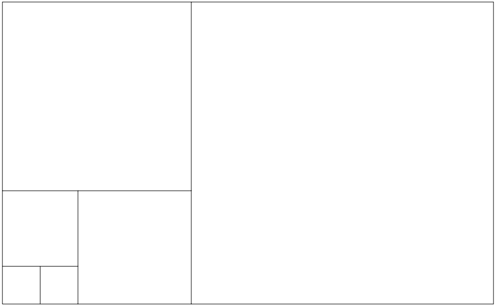
It is either covered by a domino or not. If so, the remaining number of tilings is given by $F_{m-1}F_{n-1}$, to the left and right of the domino, and if not, the remaining number of tilings is given by $F_m F_n$, to the left and the right of the line.

□

Theorem:

$F_0^2 + F_1^2 + \cdots + F_n^2 = F_n \cdot F_{n+1}$

Proof:



What is the area of a $F_n \times F_{n-1}$ rectangle?

Definition:

- Denote $n! := n \cdot (n-1) \cdot \dots \cdot 1$. It counts the number of ways to order n people in a line.
- Denote
$$\binom{n}{k}$$
for the number which counts the number of ways to choose k (distinguishable) things from a collection of n things, where the order you choose the things does *not* matter.

In combinatorics, this is given as an *axiomatic* definition. You may or may not recall the formula from your algebra courses, but let's prove it now!

Theorem:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Proof: We will instead prove the following identity by *double counting*:

$$\binom{n}{k} k!(n-k)! = n!$$

Recall $n!$ counts the number of ways to order people in a line. Now, let's ask the question, "How many ways are there to choose k people from a group of n, order them first in a line, then order the remaining $(n-k)$ in the back of the line?"

1. First, we must choose k from the n.
2. Then, we order those k.
3. Finally, we order the remaining $(n-k)$ and put them on the back.

From the definitions, this is $\binom{n}{k}k!(n-k)!$. But in doing this we've created a total order of the line! We've counted how to do the same thing in two different ways - hence the identity is proven. □

Let's now consider *Pascal's triangle*, which has in the nth row the values of $\binom{n}{k}$.

1					
1	1				
1	2	1			
1	3	3	1		
1	4	6	4	1	
1	5	10	10	5	1

- The value anywhere in Pascal's Triangle is equal to the sum of the values above and to the left. Precisely, $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.
- The sum across the nth row is 2^n. Why? How many ways are there to choose some number of things from n total things?

What about the sum across the diagonals running up and to the right?

$$
\begin{array}{cccccccccc}
 & 1 & & & & & & \\
 & & 1 & & & & & \\
1 & & & 2 & & & & \\
 & & & & 3 & & & \\
1 & 1 & & & & 5 & & \\
 & & & & & & 8 & \\
1 & 2 & 1 & & & & & 13 \\
1 & 3 & 3 & 1 & & & & & \ddots \\
1 & 4 & 6 & 4 & 1 & & & \\
1 & 5 & 10 & 10 & 5 & 1 & & \\
1 & \cdots & & & & & &
\end{array}
$$

The Fibonaccis?

We can represent theses diagonal sums with:

$$\binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \dots$$

(where we assume $\binom{n}{k} = 0$ when $k > n$)

Theorem:

$\binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots = F_n$

Proof: There are F_n ways to tile a $1 \times n$ grid. Now, how many tilings are there where exactly k dominoes?

- $k = 0$ - there is only 1, the tiling consisting of all squares!
- $k = 1$ - since there is 1 domino, there must in total be $n - 1$ total squares/dominoes. We just need to choose one of those to be a domino, there are $\binom{n-1}{1}$ ways of doing this.
- $k = 2$ - since there are 2 dominoes, there must be in total $n - 2$ squares/dominoes. We need to choose two of the tiles to be dominoes, and there are $\binom{n-2}{2}$ ways of doing this.
- k general - now, there will be $n - k$ squares/dominoes, and we need to choose k of them to be dominoes. There are $\binom{n}{k}$ ways of doing this.

If we consider the sum of possibilities over all k, we're just asking how many ways are there to tile the $1 \times n$ grid, with any amount of dominoes!

□

Thanks for listening!
Questions?