On endotrivial complexes and the generalized Dade group

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Conventions & Notation

- G is a finite group.
- k is a field of characteristic $p > 0$.
- $s_p(G)$ denotes the set of p-subgroups of G.
- $\mathrm{Syl}_p(G)$ denotes the set of Sylow p-subgroups of G.
- All kG-modules are finitely generated.
- ${}_{kG}\mathbf{triv}$ is the category of f.g. p-permutation kG-modules.

Preliminaries

Definition

- A kG-module M is a permutation module if $M \cong k[X]$ for some G-set X.
- A kG-module M is a p-permutation module if for all $P \in s_p(G)$, $\operatorname{res}^G_P M$ is a permutation module, or equivalently, if M is a direct summand of a permutation module.

Definition

- For any $P \in s_p(G)$, the Brauer construction is an additive functor $: -(P) : {}_{kG}\textbf{mod} \to {}_{k[N_G(P)/P]}\textbf{mod}$.
- The Brauer construction restricts to a functor $-(P) : {}_{kG}\textbf{triv} \to {}_{k[N_G(P)/P]}\textbf{triv}$ which is multiplicative, i.e. for $M, N \in {}_{kG}\textbf{triv}$, we have a natural isomorphism

$$(M \otimes_k N)(P) \cong M(P) \otimes_k N(P).$$

Think of the Brauer construction as a "P-fixed-points-on-G-sets" functor. Indeed, $k[X](P) \cong k[X^P]$.

Endotrivial and endopermutation modules

Motivation one: endopermutation and endotrivial modules!

Definition

Let M be a kG-module.

1. M is endotrivial if and only if

$$M^* \otimes_k M \cong k \oplus N,$$

   for some projective kG-module N.
2. M is endopermutation if and only if $M^* \otimes_k M$ is a permutation module.

Goals: classify all endotrivial and endopermutation kG-modules.

Endotrivial modules

The group $(T_k(G), \otimes_k)$ parametrizes endotrivial modules.

$$T_k(G) := \{[M] \in \mathsf{stmod}(kG) \mid M \text{ is endotrivial}\}.$$

Known results

- $T_k(G)$ is finitely generated abelian. (Puig '90, CMN '06)
- $T_k(G)$ is determined for p-groups. (CT '00-'04)
- $T_k(G)$ is determined for finite groups of Lie type (CMN '06)
- ...and many, many more!

Completely determining $T_k(G)$ for all groups remains open.

Relatively endotrivial modules

Lassueur generalized endotriviality in her Ph.D. thesis to the relative projectivity setting.

Definition

Let V and M be kG-modules.

1. M is V-projective if M is a direct summand of $V \otimes_k N$ for some kG-module N.
2. (Lassueur '11) M is relatively V-endotrivial if $M^* \otimes_k M \cong k \oplus N$ for some V-projective kG-module N.

The group $(T_V(G), \otimes_k)$, parametrizes the relatively V-endotrivial modules.

$$T_V(G) := \{[M] \in \text{stmod}(V) \mid M \text{ is relatively } V\text{-endotrivial}\}.$$

The Dade group

Assume G is a p-group.

Say an endopermutation kG-module M is capped if M has a direct summand with vertex G. The Dade group $D_k(G)$ parameterizes the capped endopermutation kG-modules.

Known results

$D_k(G)$ is completely classified for all p-groups G. (Bouc, Carlson, Dade, Thévenaz, Yalçin, et. al)

This classification uses Bouc's theory of *rational p-biset functors*, an extension of Mackey functors for p-groups.

The generalized Dade group

What if G is an arbitrary finite group?

Linckelmann/Mazza and Lassueur, using separate methods, generalized the Dade group for finite groups.

Definition (Lassueur '13)

Set

$$V(\mathcal{F}_G) := \bigoplus_{P \in s_p(G) \setminus \mathrm{Syl}_p(G)} k[G/P].$$

- An endo-p-permutation kG-module is strongly capped if it is $V(\mathcal{F}_G)$-endotrivial.
- $D_k(G) \leq T_{V(\mathcal{F}_G)}(G)$ is the subgroup of $T_{V(\mathcal{F}_G)}(G)$ generated by equivalence classes of strongly capped endotrivial kG-modules.

If G is a p-group, we recover the classical Dade group.

Splendid Rickard equivalences

Motivation two: splendid Rickard equivalences and Broué's abelian defect group conjecture!

Definition

Let G, H be finite groups and let A, B be block algebras of kG, kH respectively. A splendid Rickard equivalence for A and B is a chain complex X of p-permutation (A, B)-bimodules with *twisted diagonal vertices* satisfying:

1. $X \otimes_B X^* \simeq A[0]$ as chain complexes of (A, A)-bimodules.
2. $X^* \otimes_A X \simeq B[0]$ as chain complexes of (B, B)-bimodules.

Broué's abelian defect group conjecture

If A is a block algebra with abelian defect groups, there exists a splendid Rickard equivalence between A and its Brauer correspondent.

Constructing these complexes is very difficult. We want more examples to understand them better!

Endotrivial complexes

Definition

- A bounded chain complex $C \in Ch^b({}_{kG}\textbf{triv})$ is endotrivial if

$$\mathrm{End}_k(C) \cong C^* \otimes_k C \simeq k[0],$$

  i.e. $C^* \otimes_k C \cong k \oplus D$ for some contractible chain complex D.
- Let $\mathcal{E}_k(G)$ denote the set of homotopy classes of endotrivial kG-complexes. $(\mathcal{E}_k(G), \otimes_k)$ forms an abelian group.

Goal: classify all endotrivial complexes, i.e. determine the structure of $\mathcal{E}_k(G)$.

Examples

Let $p = 2$. Examples of endotrivial complexes:

1. $kC_2 \twoheadrightarrow k$
2. Let $n \geq 3$ and let H_1, H_2 be noncentral, nonconjugate subgroups of D_{2^n} of order 2.

$$
\begin{array}{ccccc}
 & & k[D_{2^n}/H_1] & & \\
 & \nearrow & & \searrow & \\
kD_{2n} & & \oplus & & k \\
 & \searrow & & \nearrow & \\
 & & k[D_{2^n}/H_2] & &
\end{array}
$$

Here, the homomorphisms are induced from G-set homomorphisms.

Splendid Rickard equivalences

Endotrivial complexes induce splendid Rickard autoequivalences!

Theorem

Let C be an endotrivial complex of kG-modules. Let $\phi \in \mathrm{Aut}(G)$ and set

$$\Delta_\phi G = \{(\phi(g), g) \in G \times G \mid g \in G\} \cong G.$$

$\mathrm{ind}_{\Delta_\phi G}^{G\times G} C$*, regarded as a chain complex of* (kG, kG)*-bimodules, is a splendid Rickard autoequivalence of* kG*.*

Ongoing work: using these and the trivial source ring (the Grothendieck ring of ${}_{kG}$**triv**) to study the relationship between splendid Rickard equivalences and p-permutation equivalences.

Relatively endotrivial complexes

We have a relative projectivity setting for endotrivial complexes.

Definition

Let V be a kG-module (possibly 0).

- A bounded chain complex $C \in Ch^b({}_{kG}\textbf{triv})$ is V-endosplit-trivial if

$$C^* \otimes_k C \simeq (k \oplus N)[0],$$

  where N is a V-projective kG-module.
- Two V-endosplit-trivial complexes are *equivalent* if they contain isomorphic indecomposable V-endosplit-trivial complexes as direct summands. $\mathcal{E}_k^V(G)$, the collection of all equivalence classes of V-endosplit-trivial complexes, forms an abelian group under $\otimes_k$.

Notes:

- Letting $V = 0$ recovers endotrivial complexes.
- V-endosplit-trivial complexes are equivalently *endosplit p-permutation resolutions* of V-endotrivial modules.

Homology

- If C is V-endosplit-trivial, then there is a *unique* $i \in \mathbb{Z}$ for which $H_i(C) \neq 0$ by the Künneth formula.
- For any $P \in s_p(G)$, the Brauer construction induces a group homomorphism $-(P) : \mathcal{E}_k^V(G) \to \mathcal{E}_k^{V(P)}(N_G(P)/P)$.

Theorem

Let $C \in Ch^b({}_{kG}\textbf{triv})$. The following are equivalent:

- *C is endotrivial.*
- *For every $P \in s_P(G)$, $C(P)$ has nonzero homology in exactly one degree, with that homology having k-dimension 1.*

h-marks

Definition

- If C is a V-endosplit-trivial complex and $P \in s_p(G)$, let $h_C(P)$ denote the degree in which $C(P)$ has nontrivial homology. Say $h_C(P)$ is the h-mark of C at P.
- Denote the group of $\mathbb{Z}$-valued class functions on p-subgroups of G by $C(G,p)$. $h_C \in C(G,p)$.

Question: How much do "local" homological properties, like the h-marks, determine the structure of an endotrivial complex?

Answer: Almost entirely!

The h-mark homomorphism

Let $S \in \mathrm{Syl}_p(G)$. $T_V(G,S) \leq T_V(G)$ is the subgroup of p-permutation V-endotrivial modules.

Theorem

$$h : \mathcal{E}_k^V(G) \to C(G,p)$$
$$[C] \mapsto h_C$$

is a well-defined group homomorphism, with $\ker h$ *the torsion subgroup of* $\mathcal{E}_k^V(G)$*,*

$$\{M[0] \mid M \text{ is an indecomposable } p\text{-permutation } V\text{-endotrivial module}\} \cong T_V(G,S).$$

If $V = V(\mathcal{F}_G)$*,* h *is surjective.*

In particular, $\mathcal{E}_k^V(G)$ is finitely generated with $\mathbb{Z}$-rank bounded by the number of conjugacy classes of p-subgroups of G.
We call h the h-mark homomorphism.

Extracting homology

Since homology of a V-endosplit-trivial complex is nonzero in only one degree, we can extract it!

We obtain a well-defined homomorphism

$$\begin{aligned}\mathcal{H} : \mathcal{E}_k^V(G) &\to T_V(G)\\ [C] &\mapsto [H_{h_1(C)}(C)]\end{aligned}$$

A short exact sequence

In the case of $V = V(\mathcal{F}_G)$, we can completely characterize the kernel and image of $\mathcal{H}$. Define $\mathcal{TE}_k(G) \leq \mathcal{E}_k(G)$ as follows:

$$\mathcal{TE}_k(G) = \{[C] \in \mathcal{E}_k(G) \mid \mathcal{H}(C) = [k]\}.$$

Theorem

We have a short exact sequence of abelian groups

$$0 \to \mathcal{TE}_k(G) \to \mathcal{E}_k^{V(\mathcal{F}_G)}(G) \xrightarrow{\mathcal{H}} T_{V(\mathcal{F}_G)}(G, S) + D_k^{\Omega}(G) \to 0,$$

where $D_k^{\Omega}(G) \leq D_k(G)$ is the subgroup of $D_k(G)$ generated by relative syzygies, i.e. kernels of the augmentation homomorphism $kX \to k$ for some G-set X.

If G is a p-group, the short exact sequence simplifies as follows:

Theorem

Let G be a p-group. We have a short exact sequence of abelian groups

$$0 \to \mathcal{E}_k(G) \to \mathcal{E}_k^{V(\mathcal{F}_G)}(G) \to D^{\Omega}(G) \to 0.$$

Borel-Smith functions

The group of class functions $C(G)$ has a subgroup $C_b(G)$, the subgroup of Borel-Smith functions. These relate to homotopy representations of the sphere.

Theorem (Bouc-Yalçin '07)

Let G be a p-group. There is a short exact sequence

$$0 \to C_b(G) \to C(G) \xrightarrow{\Psi} D^\Omega(G) \to 0,$$

where Ψ is the Bouc homomorphism. Moreover, this is a short exact sequence of rational p-biset functors.

This short exact sequence is compatible with ours via h-marks!

Theorem

Let G be a p-group. We have an isomorphism of short exact sequences.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E}_k(G) & \hookrightarrow & \mathcal{E}_k^{V(\mathcal{F}_G)}(G) & \xrightarrow{\mathcal{H}} & D^\Omega(G) & \longrightarrow & 0 \\ & & \downarrow h & & \downarrow h & & \downarrow = & & \\ 0 & \longrightarrow & C_b(G) & \hookrightarrow & C(G) & \xrightarrow{\Psi} & D^\Omega(G) & \longrightarrow & 0 \end{array}$$

Results for p-groups

Corollary

1. *Let G be a p-group and let $f \in C(G)$ be a class function. f is the h-mark function of an endotrivial complex of kG-modules if and only if f is a Borel-Smith function.*
2. *We may assign rational p-biset functor structure to $\mathcal{E}_k$ via transport. Restriction, inflation, and deflation are all what we expect, but induction is* **not** *tensor induction.*
3. *Given any p-permutation autoequivalence γ of kG, there exists a splendid Rickard autoequivalence X of kG for which $\Lambda(X) = \gamma$.*

Questions:

1. Can we describe induction functorially?
2. Given a Borel-Smith function, can we give an explicit construction of an endotrivial complex without relying on taking direct summands?

Results for non-p-groups

Previously, we determined the image of the map induced by restriction to a Sylow p-subgroup.

Theorem

Let G be a finite group and $S \in \mathrm{Syl}_p(G)$.

$$\mathrm{res}^G_S : \mathcal{E}_k(G) \to \mathcal{E}_k(S)^{\mathcal{F}}$$

is surjective, where $\mathcal{E}_k(S)^{\mathcal{F}} \leq \mathcal{E}_k(S)$ is the fusion-stable subgroup of $\mathcal{E}_k(S)$, consisting of elements $[C] \in \mathcal{E}_k(S)$ for which $h_C(P) = h_C(Q)$ for all G-conjugate $P, Q \leq S$.

Corollary

Let G be a p-group and let $f \in C(G,p)$ be a class function. f is the h-mark function of an endotrivial complex of kG-modules if and only if f is a fusion-stable Borel-Smith function.

Questions: Can we give explicit constructions of the representatives of $\mathcal{E}_k(G)$? (Seems harder!)

Thank you!!

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