What is representation theory?

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Representation theory

What is representation theory?

Representation theory is a subfield of mathematics which uses linear algebra to study algebraic objects.

Representations are ubiquitous in mathematics and beyond!

- The Langlands program, one of the largest open programs of research in mathematics, relates representations to "automorphic forms."
- Chemists, physicists, and engineers use representation theory in their work. (more on this later...)

What is a group?

First, what is a group?

A group is a set G which comes with a binary operation, that is, a function $G \times G \to G$, the group law.

Loosely, you can "multiply" group elements together to get a new group element. Given $g, h \in G$, $g \cdot h \in G$.

A group is **finite** if its underlying set is finite. For this talk, we will mostly consider finite groups.

What is a group?

Two ways I personally think about groups:

1. A group can be a generalization of a number system. Examples:
    - The integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ (or rationals $\mathbb{Q}$, reals $\mathbb{R}$, or complex numbers $\mathbb{C}$) with group law given by addition.
    - The nonzero real numbers $\mathbb{R} \setminus \{0\}$ (or nonzero rationals $\mathbb{Q} \setminus \{0\}$ or nonzero complex numbers $\mathbb{C} \setminus \{0\}$), with group law given by multiplication.
    - The integers modulo m, $\mathbb{Z}/m\mathbb{Z}$ with group law given by addition.
2. A group can describe the **symmetries** of, or ways one can act on, a mathematical object. Examples:
    - The ways you can twist a Rubik's cube, with group law given by composing twists.
    - The ways you can permute the numbers $1 - n$, with group law given by composing permutations.
    - The ways you can rotate a cube (or some object) in space which preserve its symmetry, with group law given by composing rotations.

What is a representation?

What is a representation?

A philosophy: say we have some mathematical thing, and we wish to understand its properties.

- Idea 1: study the thing directly and see what we can figure out.
- Idea 2: study the objects the thing acts upon (sets, geometric objects, vector spaces for instance), and use this data to determine qualities about our thing.

Such an object is a **representation** of our thing!

Mathematicians study the representation theory of finite groups, Lie groups, Lie algebras, or other stranger things!

What is a representation of a finite group?

What is a *representation of a finite group*?

Disclaimer: For this talk, we'll focus on representations over the *complex numbers* $\mathbb{C}$. But in general, one can consider representations over any field, and the theory can get quite different - and quite ugly!

From here on, assume all our vector spaces are over $\mathbb{C}$. Recall that *linear maps* between vector spaces correspond to *matrices* - tables of numbers (in $\mathbb{C}$) we can add and multiply.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Figure: The 2×2 identity matrix, which corresponds to the identity map.

What is a representation of a finite group?

We have two *equivalent* notions. Given a finite group G, a representation of G is:

1. a "well-behaved" map (a *group homomorphism*)

$$\rho : G \to \mathrm{Aut}(V),$$

   where V is some vector space with finite dimension d.

   That is, for every group element $g \in G$, we get an linear map $V \to V$, i.e. a $d \times d$ matrix, and group multiplication corresponds to matrix multiplication.
2. a finite dimension vector space which G acts upon in a "nice" way. We call these vector spaces $\mathbb{C}G$-**modules.**

How are these equivalent? The map in (1) tells us exactly what the nice action in (2) is!

A nice example

Let $G = S_3$, the **symmetric group** on 3 elements. S_3 is the set of ways to permute, or reorder, the elements of a 3-element set such as $\{a, b, c\}$. S_3 has $3! = 6$ elements,

$$S_3 = \{\text{id}, (12), (13), (23), (123), (132)\}.$$

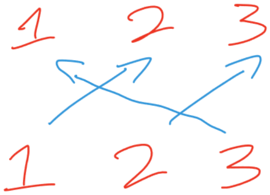


Figure: The element (123).

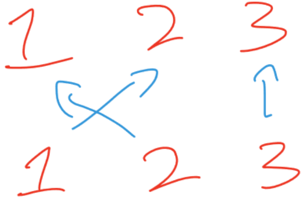


Figure: The element (12).

A nice example

The group law of S_3 is given by composing permutations.

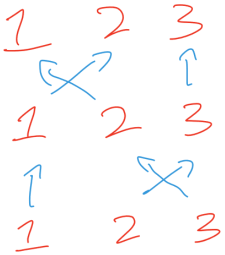


Figure: $(12) \circ (23) = (123)$.

S_3 acts on ordered sets by permuting indices.

- $(123) \cdot \{a, b, c\} = \{c, a, b\}$
- $(12) \cdot \{a, b, c\} = \{b, a, c\}$

A nice example

Let V be a vector space with dimension 3. S_3 has the following natural action on V: given $v = (x, y, z) \in V$, S_3 permutes its entries, creating a new vector. For example:

- $(12) \cdot (x, y, z) = (y, x, z)$.
- $(123) \cdot (x, y, z) = (z, x, y)$.

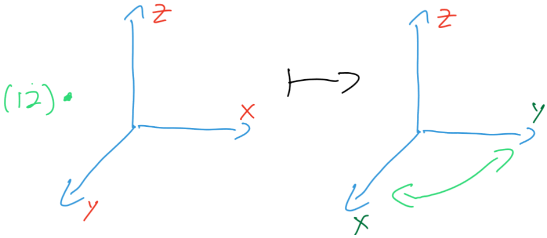


Figure: The action of $(12) \in S_3$ on V.

A nice example

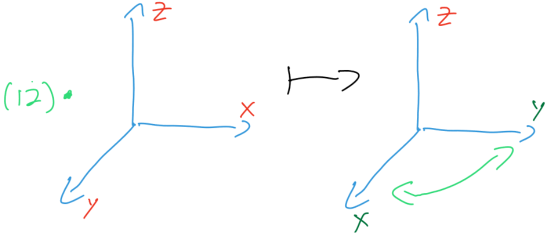


Figure: The action of $(12) \in S_3$ on V.

V is a representation of S_3, the **natural permutation representation** of S_3. The corresponding map $G \to \mathrm{Aut}(V)$ is:

$$(12) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad (123) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Subrepresentations

We can consider **subrepresentations** of representations. We'll introduce these through $\mathbb{C}G$-module language.

A **submodule** W of a $\mathbb{C}G$-module V is a vector subspace $W \subseteq V$ which is *closed under G-action*, i.e. for all $g \in G$ and $w \in W$, $g \cdot w \in W$.

For example, the *natural permutation representation* V of S_3 has the following subrepresentations:

- $W_1 = \{(x, x, x) \in V\}$. This is a 1-dimensional subspace.
- $W_2 = \{(x, y, z) \in V \mid x + y + z = 0\}$. This is a 2-dimensional subspace.

In fact, $V = W_1 \oplus W_2$, that is, every element of v can be written **uniquely** as $v = w_1 + w_2$ where $w_1 \in W_1$ and $w_2 \in W_2$.

Maschke's theorem

Theorem (Maschke)

Let V be a $\mathbb{C}G$-module. Given any $\mathbb{C}G$-submodule $W \subset V$, there exists another $\mathbb{C}G$-submodule $W' \subset V$ for which $W \oplus W' = V$.

Corollary

Every representation decomposes into a direct sum of **irreducible** *representations (or* **irreps***), that is, representations whose only subrepresentations are 0 and itself.*

To understand the representation theory of G, it is of critical importance to classify the *irreps* of G, since they are the building blocks of any representation!

Irreducible representations

One may use **character theory** to prove the following about irreducible representations of a finite group G.

1. The number of irreps is *finite*, and is equal to the number of conjugacy classes of G.
2. Given a representation V of G and sufficient knowledge about the irreps of G, one may explicitly compute the number of times each irrep occurs in a direct sum decomposition of V.
3. If $S_1, \ldots, S_k$ are the irreps of G, then

$$(\dim S_1)^2 + \cdots + (\dim S_k)^2 = |G|.$$

4. Given any irrep S of G, $\dim S$ divides $|G|$.

What is this all good for?

How does representation theory show up in other fields?

First, some finite group theory applications.

Solvability: we have a notion of a *solvable* group. This corresponds (via Galois theory) to solvability of certain polynomial equations by taking radicals, i.e. nth square roots.

Theorem (Burnside)

Let G be a finite group. If $|G| = p^a q^b$, where p, q are primes and a, b are non-negative integers, then G is solvable.

What is this all good for?

Frobenius groups: a group G is a *Frobenius group* if there exists a nontrivial subgroup $H \leq G$ such that for all $g \in G \setminus H$,

$$H \cap gHg^{-1} = \{1\}.$$

Theorem (Frobenius)

Let G be a Frobenius group with $H \leq G$ satisfying the above condition. Then the subset $N \subseteq G$ defined by

$$N := G \setminus \left(\bigcup_{g \in G} {}^{g}H \setminus \{1\} \right).$$

is a normal subgroup of G, satisfying $HN = G$ and $H \cap N = \{1\}$.

Non-mathematical applications

Representations are even important outside of mathematics!

Many physical objects exhibit physical symmetries and therefore can be considered as representations!

- **Particle Physics:** the set of quantum states of a particle forms a vector space. Particles exhibit physical symmetry, therefore the set of states of a particle forms a representation. Decomposing such a representation determines the decomposition of a particle into fundamental particles!
- **Chemistry:** some molecules exhibit physical symmetries. The vibrational modes of certain symmetric molecules can be viewed as irreps of their groups of symmetries, and can be deduced by solving certain linear differential equations!
- **Engineering:** Fourier analysis is a special case of the representation theory of the 1-sphere S^1! (i.e. the symmetries of a circle)

Symmetric groups

Let's take a look at the representation theory of the **symmetric group** on n elements, S_n.

Recall S_n is the set of *permutations*, or reorderings, of an n-element set $\{1, \ldots, n\}$, and forms a group by composition.

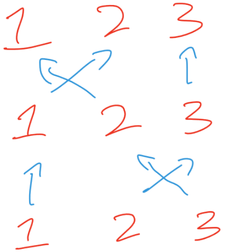


Figure: $(12) \circ (23) = (123) \in S_3$.

Partitions

A **partition** of n is a sum $\lambda : a_1 + \cdots + a_k = n$, with each a_i a positive integer and

$$a_1 \geq a_2 \geq \cdots \geq a_k.$$

The partitions of $n = 5$ are:

- 5
- $4 + 1$
- $3 + 2$
- $3 + 1 + 1$
- $2 + 2 + 1$
- $2 + 1 + 1 + 1$
- $1 + 1 + 1 + 1 + 1$

Young diagrams

We can represent a partition λ of n diagramatically.

A Young diagram is a collection of boxes, organized in *left-justified* rows and columns, with each row length *non-increasing.*

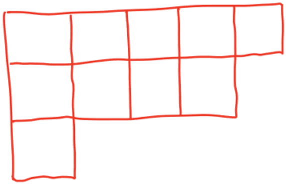


Figure: A Young diagram of shape $\lambda : 5 + 4 + 1 = 10$.

We have a *bijection* between the set of partitions of n and the set of Young diagram with n boxes. Young diagram and partitions contain the same information!

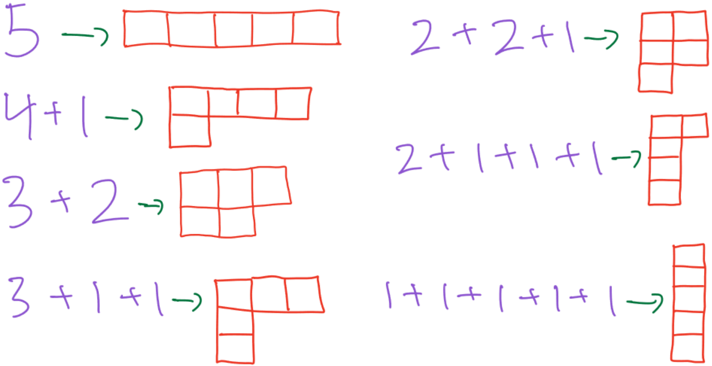


Figure: Young diagrams and partitions for $n = 5$.

Young tableau

Given a Young diagram of shape λ with n boxes, we form a Young tableau by putting the numbers $1 - n$ in each box once.

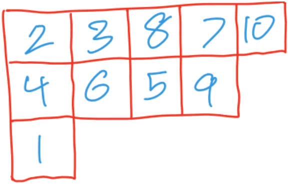


Figure: A Young tableau of shape $\lambda : 5 + 4 + 1 = 10$.

Given any partition λ of n, there are $n!$ Young tableau with shape λ.

Young tableau

Given a fixed Young diagram with shape λ, S_n *acts* on the set of Young tableau of shape λ by permuting entries.

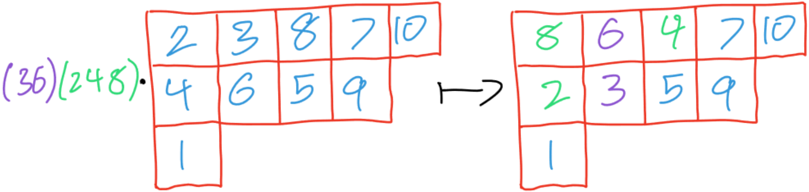


Figure: $(36)(248)$ acting on a Young tableau.

Tabloids

We say two Young tableau are **similar**, denoted $\sim$, if their entries in each row are the same, up to reordering.

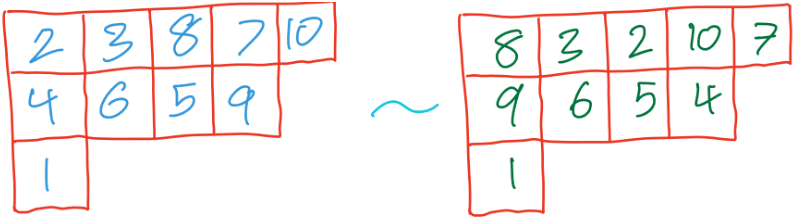


Figure: Two similar Young tableau.

Tabloids

The collection of all *similar* Young tableau forms a tabloid. The S_n action on the set of all Young tableau descends to an action on the set of all tabloids!

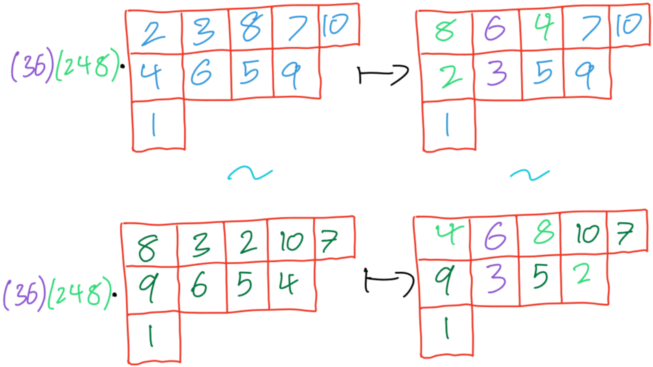
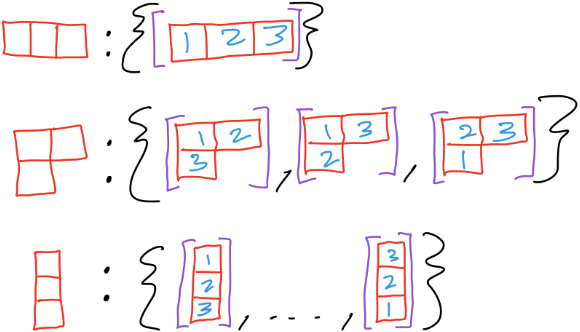


Figure: The S_n-action on Young tableau preserves similarity.

Tabloids

Given a Young tableau T, denote its corresponding tabloid by $[T]$. Given the following three partitions of 3, the set of tabloids are as follows:



Notice the number of tabloids depends on the shape λ, unlike the number of Young tableau.

Specht modules

Given a partition λ of n, the vector space W_λ with basis indexed by the tabloids of shape λ is naturally a representation of S_n, with S_n permuting indices.

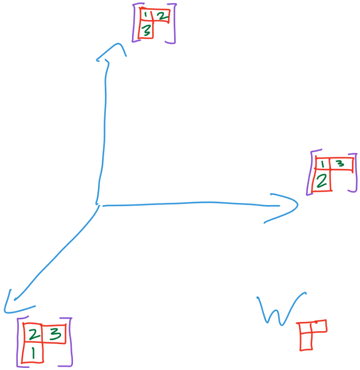


Figure: W_λ for $\lambda : 2 + 1 = 3$

Specht modules

Given any partition λ of n, W_λ has a distinguished subrepresentation V_λ, the Specht module of the partition λ.

Theorem

The set $\{V_\lambda\}$ as λ runs through all partitions of n forms a complete set of irreps of S_n.

Therefore, we have a constructive bijection:

$$\{\text{partitions of } n\} \xrightarrow{\cong} \{\text{irreps of } S_n\}$$
$$\lambda \mapsto V_\lambda$$

Specht modules

What do these Specht modules look like? For S_3:

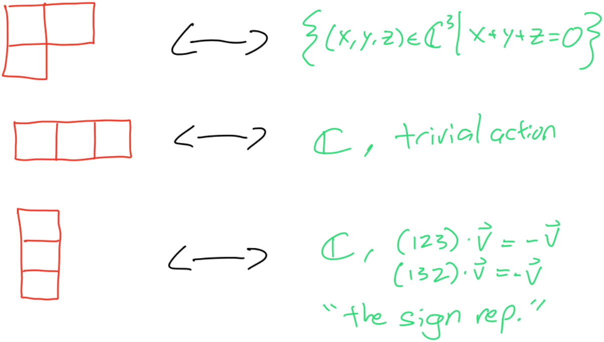


Figure: The Specht modules of S_3.

The hook length formula

Moreover, given a partition λ, we can determine the dimension of the Specht module V_λ purely combinatorially, via the *hook length formula*. Given a box in a Young diagram λ, its **hook length** is the number of boxes to the right plus the number of boxes below plus one.

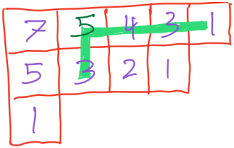


Figure: The hook lengths for $\lambda : 5 + 4 + 1 = 10$.

Theorem (Hook length formula)

If the hook lengths of each box in λ are $h_1, \ldots, h_n$, then

$$d_\lambda = \frac{n!}{h_1 \cdots h_n}$$

is the dimension of V_λ.

A nifty application

Let $\lambda : a_1 + \cdots + a_k = n$ be a partition of n.

Put a_1 ones, a_2 twos, and so on, into an urn, and draw from the urn without replacement until the urn is empty.

Theorem (Diaconis)

The chance that at every stage of the drawing, the number of ones drawn ≥ number of twos drawn ≥ ⋯ ≥ the number of ks drawn is

$$d_\lambda \cdot \frac{\prod_{i=1}^{k} a_i!}{n!}.$$

What if our field isn't $\mathbb{C}$?

What if our base field isn't $\mathbb{C}$, or in particular has positive characteristic?

For example, the integers modulo p, $\mathbb{Z}/p\mathbb{Z}$, form a field of positive characteristic, since if we add 1 to itself p times, we reach 0.

The representation theory of finite groups changes drastically when we replace $\mathbb{C}$ with k, a field of positive characteristic.

- In general, **Maschke's theorem no longer holds.**
- Therefore, representations no longer decompose into irreps.

We consider indecomposable kG-modules, modules which do not decompose into smaller direct sums, but may still contain nontrivial submodules.

Indecomposables

Instead of trying to classify the *irreps*, we can try to classify the indecomposables.

This ends up being a **much harder** question.

- The number of irreps is still finite and can be determined explicitly.
- However, for almost all groups, the number of indecomposables is infinite.
- Moreover, for almost all groups, classifying all indecomposables would entail classifying all indecomposables for every other group, as well as classifying all indecomposables of other algebraic objects!

Classifying indecomposables is **too hard** of a problem! We need to ask different questions (which are beyond the scope of this talk).

Representation theory of S_n in positive characteristic

How does the representation theory of the symmetric groups change in positive characteristic?

Given a partition λ of n, we can still define the kG-modules W_λ and the Specht module V_λ using the same constructions.

However, V_λ is no longer necessarily an *irrep*. However, we can use module theory to obtain an irrep, and all irreps can be constructed in this way.

Understanding the Specht modules in positive characteristic (such as determining which are irreducible, their endomorphism rings, and so on) remains elusive!

What can we ask?

In modular representation theory, representations can be further divided into separate **blocks**, leading to **block theory**.

A few questions and open problems:

- **Donovan's conjecture**: Are there finitely many blocks with a given *defect group*?
- **Broué's abelian defect group conjecture**: Can we relate the representation theory of a block to a corresponding "local" block?
- **Alperin's weight conjecture**: A counting conjecture about the number of irreps belonging to a block.

Thank you!

Thanks for listening!

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