2.3: Bisets

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Day 1

2.3 Bisets

Definition 2.3.1. Let G, H be groups. Then an (H, G)-biset is a left $(H \times G^{op})$-set. Equivalently, an (H, G)-biset U is both a left H-set and a right G-set, such that the H-action and the G-action commute, i.e.

$$(h \cdot u) \cdot g = h \cdot (u \cdot g).$$

Hence we may write $h \cdot u \cdot g$ or hug without ambiguity.

Remark 2.3.2. We can consider disjoint unions or products of bisets as before. If U and V are (H, G)-bisets, then a biset homomorphism $f : U \to V$ satisfies $f(h \cdot u \cdot g) = h \cdot f(u) \cdot g$. If U is a (H, G)-biset, then the set $(H \times G^{op})\backslash U$ is called **the set of** (H, G)**-orbits on** U, denoted $H\backslash U/G$. As before, the biset U is **transitive** if $H\backslash U/G$ has cardinality 1, or equivalently, there exists $(h, g) \in H \times G$ such that $h \cdot u \cdot g = v$.

Example 2.3.3 (Identity Bisets)**.** If G is a group, then the set G is a (G, G)-biset for the left and right actions of G on itself by multiplication. The biset is called **the identity** (G, G) **biset** and is denoted by Id_G. More generally, if H is a subgroup of G, then the set G/H is a $(G, N_G(H)/H)$-biset, and $H\backslash G$ is a $(N_G(H)/H, G)$-biset. More precisely for G/H: the action is as follows: for $hH \in N_G(H)/H$ and $g_1, g_2 \in G$,

$$g_1 \cdot g_2H \cdot hH = g_1g_2hH.$$

This is well-defined on the right: this is valid multiplication since $hH = Hh$, and for $h_1H = h_2H$, we have $h_1h = h_2$ for some $h \in H$. Then,

$$gh_2H = gh_1hH = gh_1H.$$

Lemma 2.3.4. 1. If L is a subgroup of $H \times G$, then the set $(H \times G)/L$ is a transitive (H, G)-biset for the actions defined by:

$$\forall h \in H, \forall (b, a)L \in (H \times G)/L, \forall g \in G, h \cdot (b, a)L \cdot g = (hb, g^{-1}a)L.$$

2. If U is an (H,G)-biset choose a set $[H\backslash U/G]$ of representations of (H,G)-orbits on U. Then there is an isomorphism of (H,G)-bisets

$$U \cong \bigsqcup_{u \in [H\backslash U/G]} (H \times G)/L_u,$$

where $L_u = (H,G)_u$ is the stabilizer of u in $H \times G$, i.e. the subgroup of $H \times G$ defined by

$$(H,G)_u = \{(h,g) \in H \times G : h \cdot u = u \cdot g\}.$$

In particular, any transitive (H,G)-biset is isomorphic to $(H \times G)/L$ for some subgroup L of $H \times G$.

Proof. 1. This statement is a straightforward verification.

2. This statement follows directly from Lemma 2.2.2. One must note that since U is a (H,G)-biset, so the action of $(H \times G)$ on U is given by $(h, g^{-1}) \cdot u = h \cdot u \cdot g^{-1}$, which determines L_u. Equivalently, we could write

$$U \cong \bigsqcup_{u \in [H\backslash U/G]} (H \times G^{op})/L_u$$

where $L_u = \{(h,g) \in H \times G^{-1} : u \cdot h \cdot g = h\}$.

□

Example 2.3.5. Let $f : G \to H$ be a group homomorphism. Then the set H has a (H,G)-biset structure given by

$$h \cdot k \cdot g = hkf(g).$$

This biset is isomorphic to $(H \times G)/\Delta_f(G)$, where $\Delta_f(G)$ is the graph of f,

$$\Delta_f(G) = \{(f(g), g) : g \in G\}.$$

The bijection is given by $\phi : k \mapsto (k,1)\Delta_f(G)$. One verifies the map satisfies the right action:

$$\phi(k \cdot g) = \phi(kf(g)) = (kf(g), 1)\Delta_f(G) = (k, g^{-1})\Delta_f(G) = (k,1)\Delta_f(G) \cdot g = \phi(k) \cdot g.$$

The map has inverse

$$\psi : (h,g)\Delta_f(G) = (hf(g^{-1}), 1)\Delta_f(G) \mapsto hf(g^{-1})$$

Definition 2.3.6. Let G and H be groups. If U is an (H,G)-biset, then the opposite biset U^{op} is the (G,H)-biset equal to U as a set, with actions defined by

$$\forall g \in G, u \in U, h \in H, g \cdot u \cdot h \text{ (in } U^{op}) = h^{-1}ug^{-1} \text{ (in } U).$$

Example 2.3.7. If H is a subgroup of G, then the map $xH \mapsto Hx^{-1}$ is an isomorphism of $(G, N_G(H)/H)$-bisets from G/H to $(H\backslash G)^{op}$.

Example 2.3.8 (Opposite Subgroup)**.** If G and H are groups, and $L \leqslant H \times G$, then the **opposite subgroup** $L^{\diamond} \leqslant G \times H$ defined by

$$L^{\diamond} = \{(g,h) \in G \times H : (h,g) \in L\}.$$

With this notation, there is an isomorphism of (G,H)-bisets

$$\big((H \times G)/L\big)^{op} \cong (G \times H)/L^{\diamond}, (h,g)L \mapsto (g,h)L^{\diamond}$$

(one must verify this map is well-defined as a map of (G,H)-bisets).

Remark 2.3.9 (Elementary Bisets)**.** Let G be a group. The following bisets are fundamental:

- If $H \leqslant G$, the set G is an (H,G)-biset in the obvious way. It is denoted by Res^G_H, where Res means restriction.
- Similarly, G is a (G,H)-biset in the obvious way. It is denoted by Ind^G_H, where Ind means induction.
- If $N \trianglelefteq G$ and $H = G/N$, the set H is a (G,H)-biset, for the right action of H by multiplication, and the left action of G by projection to H, then left multiplication. It is denoted by Inf^G_H, where Ind means induction.
- Similarly, H is a (H,G)-biset in the same way as before. It is denoted by Def^G_H, where Def means deflation.
- If $f : G \to H$ is a group isomorphism, then the set H is a (H,G)-biset, for the left action of H by multiplication, and the right action of G given by taking the image in f, then multiplying on the right in H. It is denoted by $\mathrm{Iso}(f)$ of Iso^H_G if the isomorphism f is clear from context.

Composition of Bisets

Definition 2.3.11. Let G, H, K be groups, and let U be a (H, G)-biset and V a (K, H)-biset. Define the **composition of V and U** to be the set of H-orbits on the right H-action on $V \times U$, where the right action of H is given by

$$(v, u) \cdot h = (v \cdot h, h^{-1} \cdot u).$$

Denote this set by $V \times_H U$, and denote the H-orbit of $(v, u) \in V \times U$ by $(v,_H u) \in V \times_H U$. $V \times_H U$ is a (K, G)-biset for the actions defined by

$$k \cdot (v,_H u) \cdot g = (k \cdot v,_H , u \cdot g).$$

We will verify well-definedness of this action. Let $(v_1,_H , u_1) = (v_2,_H u_2) \in (V \times_H U)$, and let $k \in K, g \in G$. There exists $h \in H$ such that

$$(v_1, u_1) \cdot h = (v_1 \cdot h, h^{-1} \cdot u_1) = (v_2, u_2),$$

so $v_1 \cdot h = v_2$ and $h^{-1} \cdot u_1 = u_2$. Then

$$(k \cdot v_1, u_1 \cdot g) = (k \cdot v_2 \cdot h, h^{-1} \cdot u_2 \cdot g) = (k \cdot v_1, u_2 \cdot g) \cdot h,$$

and hence $(k \cdot v_1,_H u_1 \cdot g) = (k \cdot v_2,_H u_2 \cdot g)$ as desired.

Definition 2.3.12. Let G be a group. A **section (T, S) of G** is a pair of subgroups of G such that $S \trianglelefteq T$. The **associated subquotient** of G is the factor group T/S.

Example 2.3.13 (Defres and Indinf)**.** Let G be a group and let (T, S) be a section of G (so $S \trianglelefteq T \leqslant G$). Then there is an isomorphism of $(G, T/S)$-bisets:

$$\mathrm{Ind}_T^G \times_T \mathrm{Inf}_{T/S}^T \xrightarrow{\cong} G/S$$

sending $(g,_T tS)$ to gtS. For this reason, the $(G, T/S)$-biset G/S will be denoted by $\mathrm{Indinf}_{T/S}^G$.

Let's verify this! Recall Ind_T^G is G as a (G, T)-biset, and $\mathrm{Inf}_{T/S}^T$ is T/S viewed as a (T, S)-biset, so the definition makes sense. The map is well-defined: since every element of $(g,_T tS)$ is of the form $(gt', (t')^{-1}tS)$ for some $t' \in T$, any choice of representative is sent to $gt'(t')^{-1}tS = gtS$ via the isomorphism. Moreover, the inverse map is given by $gS \mapsto (g, 1S)$, and it is straightforward to see that these maps are indeed inverse (since $(g,_T tS) = (gt,_T 1S)$). Finally, one verifies that these are $(G, T/S)$-equivariant maps.

Similarly, there is an isomorphism of $(T/S, G)$-bisets

$$\mathrm{Def}_{T/S}^T \times_T \mathrm{Res}_T^G \xrightarrow{\cong} S\backslash G,$$

sending $(tS,_T g)$ to Stg. For this reason, the $(T/S, G)$-biset $S\backslash G$ will be denoted by $\mathrm{Defres}_{T/S}^G$. The verification of this is similar to before.

Proposition 2.3.14. Let G, H, K, L be groups.

1. If U is an (H, G)-biset, if V is a (K, H)-biset, and W is an (L, K)-biset, then there is a canonical isomorphism of (L, G)-bisets

$$W \times_K (V \times_H U) \xrightarrow{\cong} (W \times_K V) \times_H U$$

given by $\big(w,_k (v,_h, u)\big) \mapsto \big((w,_K v),_H u)\big)$ for all $(w, v, u) \in W \times V \times U$.

2. If U is an (H, G)-biset and V is a (K, H)-biset, then there is a canonical isomorphism of (G, K)-bisets

$$(V \times_H U)^{op} \xrightarrow{\cong} U^{op} \times_H V^{op}$$

given by $(v,_H u) \mapsto (u,_H v)$.

3. If U and U' are (H, G)-bisets and if V and V' are (K, H)-bisets, then there are canonical isomorphisms of (K, G)-bisets

$$V \times_H (U \sqcup U') \cong (V \times_H U) \sqcup (V \times_H U')$$

$$(V \sqcup V') \times_H U \cong (V \times_H U) \sqcup (V' \times_H U).$$

The first is defined by

$$(v,_H u) \mapsto \begin{cases} (v,_H u) \in (V \times_H U) & u \in U \\ (v,_H u) \in (V \times_H U') & u \in U' \end{cases}$$

and the second follows similarly.

4. If U is an (H, G)-biset, then there are canonical (H, G)-biset isomorphisms

$$\mathrm{Id}_H \times_H U \xrightarrow{\cong} U \xleftarrow{\cong} U \times_G \mathrm{Id}_G$$

given by $(h,_H u) \mapsto h \cdot u$ and $(u,_G g) \mapsto u \cdot g$ for all $(h, u, g) \in H \times U \times G$.

The proof of this proposition is fairly straightforward, it's mostly just verifying that the defined maps are equivariant. (Note I added in the definition of the map in part 3.)

Remark 2.3.15. Assertion 1 allows for the unambiguous notation of $W \times_K V \times U$ and $(w,_K v,_H, u)$.

Definition 2.3.16. Let G, H be groups and U a (H, G)-biset.

1. If $L \leqslant H$, and $u \in U$, define

$$L^u := \{g \in G : \exists l \in L, l \cdot u = u \cdot g\} \subseteq G.$$

Then L^u is a subgroup of G. In particular, 1^u is the stabilizer of u in G, considering U as a right G-set.

2. If K is a subgroup of G, then set

$$^{u}K = \{h \in H : \exists k \in K, h \cdot u = u \cdot k\} \subseteq H.$$

Then ^{u}K is a subgroup of H. In particular, $^{u}1$ is the stabilizer of u in H, considering U as a left H-set.

Let's verify that $L^u \leqslant G$, the other side follows similarly. If $g_1, g_2 \in L^u$, then there exists $l_1, l_2 \in L$ such that $l_1 \cdot u = u \cdot g_1, l_2 \cdot u = u \cdot g_2$. First, $l_1^{-1} \cdot u = u \cdot g_1^{-1}$, so $g_1 \in L^u$. Next note $(l_1 \cdot u) \cdot g_2 = (u \cdot g_1) \cdot g_2$, but by commutativity of bisets, we have:

$$\begin{aligned}
(l_1 \cdot u) \cdot g_2 &= (u \cdot g_1) \cdot g_2 \\
l_1 \cdot (u \cdot g_2) &= u \cdot g_1 g_2 \\
l_1 \cdot (l_2 \cdot u) &= u \cdot g_1 g_2 \\
l_1 l_2 \cdot u &= u \cdot g_1 g_2
\end{aligned}$$

as desired.

Remark 2.3.17. If G is a group, if $U = \mathrm{Id}_G$, and $H \leqslant G$, then $H^u = u^{-1}Hu$, for $u \in G$, and $^{u}H = uHu^{-1}$. So the above notation is a generalization of the usual notation of conjugation of subgroups.

Let's verify this for H^u. $H^u = \{g \in G : \exists h \in H, h \cdot u = u \cdot g\}$, or equivalently, $h^u = \{g \in G : \exists h \in H, g = u^{-1}hu\}$, which indeed is precisely $u^{-1}Hu$.

Proposition 2.3.18. Let G, H be groups and let U be a (H, G)-biset.

1. If $u \in U$ and (T, S) is a section of H, then (T^u, S^u) is a section of G. If (Y, X) is a section of G, then $(^{u}Y, {}^{u}X)$ is a section of H.

2. In particular, if $u \in U$, then $1^u \trianglelefteq H^u$ and $^{u}1 \trianglelefteq {}^{u}G$, and there is a canonical group isomorphism

$$\overline{c_u} : H^u/1^u \xrightarrow{\cong} {}^{u}G/{}^{u}1,$$

defined by $\overline{c_u}(g1^u) = h^{u}1$, where $g \in H^u$ and $h \in H$ is such that $h \cdot u = u \cdot g$.

3. The stabilizer $(H, G)_u$ of u in $H \times G$ is equal to the set of pairs (h, g) in $^{u}G \times H^u$ such that $h^{u}1 = \overline{c_u}(g1^u)$.

4. The group $^{u}1 \times 1^u$ is a normal subgroup of $(H, G)_u$ and there are canonical group isomorphisms

$$^{u}G/{}^{u}1 \xleftarrow{\cong} (H, G)_u/(^{u}1 \times 1^u) \xrightarrow{\cong} H^u/1^u$$

defined by $(h, g)(^{u}1 \times 1^u) \mapsto h^{u}1$ and $(h, g)(^{u}1 \times 1^u) \mapsto g1^u$.

Proof. 1. Let $u \in U$ and (T, S) be a section of H. It is immediate from the definitions that $S^u \leqslant T^u \leqslant G$. It remains to show normality. Now if $g \in T^u$ and $g' \in S^u$, then there exist $t \in T$ and $s \in S$ such that $t \cdot u = u \cdot g$ and $s \cdot u = u \cdot g'$. We wish to show $gg'g^{-1} \in S$. We compute:

$$(u \cdot g)g'g^{-1} = (t \cdot u \cdot g')g^{-1} = t(s \cdot u \cdot g^{-1}) = ts(t^{-1} \cdot u),$$

hence by definition, $gg'g^{-1} \in S^u$ since $tst^{-1} \in S$ because $S \trianglelefteq T$. Thus, $S^u \trianglelefteq T^u$. The other half of (1) is similar to prove.

2. Assertion 1 implies $1^u \trianglelefteq H^u$, as $(H, 1)$ is clearly a section of G. Now if $g \in H^u$ and $h \in H$ satisfy $h \cdot u = u \cdot g$, then $h \in {}^uG$ by definition. Let $h' \in H$ be another element satisfying $h' \cdot u = u \cdot g = h \cdot u$. Then, we see $(h^{-1}h') \cdot u = u$, so $h' \in h^u1$ (recalling that ${}^u1$ is simply the stabilizer). Thus, the map

$$c_u : H^u \to {}^uG/{}^u1, g \mapsto h^u1 \text{ where } h \cdot u = u \cdot g$$

is well defined. We check it is a group homomorphism: if $g_1, g_2 \in H$, then $c_u(g_1)c_u(g_2) = h_1h_2{}^u1$, where $h_1 \cdot u = u \cdot g_1$ and $h_2 \cdot u = u \cdot g_2$. It follows from prior computations (2.3.16) that $h_1h_2 \cdot u = u \cdot g_1g_2$, so $c_u(g_1g_2) = h_1h_2{}^u1$, as desired.

Moreover c_u is surjective, since for any $h \in {}^uG$, there exists a $g \in G$ with $h \cdot u = u \cdot g$ by definition of ${}^uG$. Finally, the kernel of c_u is precisely 1^u: $c_u(g) = 1^u1$ if and only if $u = u \cdot g$ if and only if g stabilizes u if and only if $g \in 1^u$, so the induced isomorphism is exactly as desired.

3. Recall that $H \times G$ acts on U by $h \cdot u \cdot g = hug^{-1}$. Therefore, the stabilizer is precisely $\{(h, g) \in H \times G : h \cdot u = u \cdot g\}$. On the other hand, $h^u1 = \overline{c_u}(g1^u)$ if and only if $h \cdot u = u \cdot g$, as desired.

4. It follows from the definition of $(H, G)_u$ and (2) that ${}^u1 \times 1^u$ is normal in $(H, G)_u \leqslant H \times G$. The map $(h, g)({}^u1 \times 1^u) \mapsto h^u1$ is well-defined: suppose $(h_1, g_1)({}^u1 \times 1^u) = (h_2, g_2)({}^u1 \times 1^u)$, then $(h_1h_2^{-1}, g_1g_2^{-1}) \in ({}^u1 \times 1^u)$. Hence $h_1{}^u1 = h_2{}^u1$, as desired. It is clear the map is a group homomorphism, and the map has inverse defined by $h^u1 \mapsto (h, g)({}^u1 \times 1^u)$, where $\overline{c_u}(g1^u) = h^u1$ (checking this is well-defined is similar to before). Thus, we have the given isomorphism on the left. The isomorphism on the right follows similarly.

□

Day 2

Definition 2.3.19. Let G, H, K be groups. If $L \leqslant H \times G$, and if $M \leqslant K \times H$, set

$$M * L = \{(k, g) \in K \times G : \exists h \in H,\ (k, h) \in M \text{ and } (h, g) \in L\}.$$

$M * L$ is a subgroup of $K \times G$ - this is a straightforward verification.

Lemma 2.3.20. Let G, H, K be groups, let U be a (H, G)-biset and V a (K, H)-biset. Then if $u \in U$ and $v \in V$, the stabilizer of $(v,_H u)$ in $K \times G$ is equal to

$$(K, G)_{(v,_H u)} = (K, H)_v * (H, G)_u.$$

Proof. Suppose $(k, g) \in (K, G)_{(v,_H u)}$, that is, it satisfies $k \cdot (v,_H u) = (v,_H u) \cdot g$. Then $(kv,_H u) = (v,_H ug)$,, so there exists some $h \in H$ satisfying $(kv,_H u) = (v,_H ug) \cdot h = (vh,_H h^{-1}ug)$, so $kv = vh$ and $hu = ug$. So $(k, h) \in (K, H)_v$ and $(h, g) \in (H, G)_u$, and thus $(k, g) \in (K, H)_v * (H, G)_u$.

Conversely, if $(k, g) \in (K, H)_v * (H, G)_u$, so there exists $h \in H$ satisfying $kv = vh$ and $hu = ug$. Thus,

$$k(v,_H u) = (kv,_H u) = (vh,_H u) = (v,_H ug) = (v,_H u)g,$$

and so $(k, g) \in (K, G)_{(v,_H u)}$ as desired. □

Definition 2.3.21. Let G, H be groups and $L \leqslant H \times G$. Define:

$$\begin{aligned}
p_1(L) &= \{h \in H : \exists g \in G,\ (h, g) \in L\} \\
p_2(L) &= \{g \in G : \exists g \in G,\ (h, g) \in L\} \\
k_1(L) &= \{h \in H : (h, 1) \in L\} \\
k_2(L) &= \{g \in G : (1, g) \in L\} \\
q(L) &= L/(k_1(L) \times k_2(L))
\end{aligned}$$

With this notation, the stabilizer in $H \times G$ of the element $u = (1, 1)L$ of the biset $(H \times G)/L$ is obviously the group L.
The group H^u as defined previously is equal to the projection $p_2(L)$ of L on G: $H^u := \{g \in G : \exists h \in H,\ h \cdot (1, 1)L = (1, 1)L \cdot g\}$, and the equality only holds when $(h, g) \in L$ (remember $(1, 1)L \cdot g = (1, g^{-1})L$). Similarly ${}^uG = p_1(L)$.
The stabilizer ${}^u1$ of u in H is the group $k_1(L)$ (this follows similarly as before) and the stabilizer 1^u of u in G is the group $k_2(L)$.
The isomorphism $\overline{c_u}$ from Prop (2.3.18) is the map:

$$c_u : p_2(L)/k_2(L) \to p_1(L)/k_1(L), \quad gk_2(L) \mapsto hk_1(L) \text{ with } (h, g) \in L.$$

Prop (2.3.18.4) implies $(k_1(L) \times k_2(L)) \trianglelefteq L$, and there are canonical group isomorphisms:

$$p_1(L)/k_1(L) \cong q(L) \cong p_2(L)/k_2(L).$$

Lemma 2.3.22. Let G, H, K be groups. Let $L \leqslant H \times G$ and $M \leqslant K \times H$.

1. There are exact sequences of groups:

$$1 \to k_1(M) \times k_2(L) \xrightarrow{i} M * L \xrightarrow{\theta} \big(p_2(M) \cap p_1(L)\big)/\big(k_2(M) \cap k_1(L)\big) \to 1$$

$$1 \to k_1(M) \to k_1(M * L) \to \big(p_2(M) \cap k_1(L)\big)/\big(k_2(M) \cap k_1(L)\big) \to 1$$

$$1 \to k_2(L) \to k_2(M * L) \to \big(k_2(M) \cap p_1(L)\big)/\big(k_2(M) \cap k_1(L)\big) \to 1$$

2. There are inclusions of subgroups

$$k_1(M) \subseteq k_1(M * L) \subseteq p_1(M * L) \subseteq p_1(M)$$

$$k_2(L) \subseteq k_2(M * L) \subseteq p_2(M * L) \subseteq p_2(L)$$

Proof. 1. First note that $k_1(M) \times k_2(L) \leqslant M * L$, since any $h \in k_1(M)$ satisfies $(h, 1) \in M$ and $g \in k_2(L)$ satisfies $(1, g) \in L$. We set i to be the inclusion, it is obviously injective.

Let $(k, g) \in M * L$, so there exists $h \in H$ such that $(k, h) \in M$ and $(h, g) \in L$. Then, $h \in p_2(M)$ and $h \in p_1(L)$, and thus $h \in p_2(M) \cap p_1(L)$. Now suppose $h' \in H$ also satisfies $(k, h') \in M$ and $(h', g) \in L$. then $(1, h^{-1}h') \in M$ and $(h^{-1}h', 1) \in L$, so $h^{-1}h' \in k_2(M)$ and $h^{-1}h' \in k_1(L)$, and hence $h^{-1}h' \in k_2(M) \cap k_1(L)$. Therefore, the map $\theta : (k, g) \mapsto h\big(k_2(M) \cap k_1(L)\big)$ is a well-defined map from $M * L$ to $\big(p_2(M) \cap p_1(L)\big)/\big(k_2(M) \cap k_1(L)\big)$. It is straightforward to check it is a group homomorphism.

This morphism is surjective: if $h \in p_2(M) \cap p_1(L)$, then there exists a $k \in K$ for which $(k, h) \in M$ and there exists $g \in G$ such that $(h, g) \in L$. So $(k, g) \in M * L$, and $\theta(k, g) = h(k_2(M) \cap k_1(L))$.

We next check exactness. If $k \in k_1(M)$ and $g \in k_2(L)$, then by definition $(k, 1) \in M$ and $(1, g) \in L$. So $\theta(k, g) = 1\big(k_2(M) \cap k_1(L)\big)$, i.e. $k_1(M) \times k_2(L) \leqslant \ker\theta$. Conversely, suppose $(k, g) \in \ker\theta$. Then there exists $h \in k_2(M) \cap k_1(L)$ for which $(k, h) \in M$ and $(h, g) \in L$. Since $h \in k_2(M)$, $(1, h) \in M$, and $(k, 1) = (k, h)(1, h)^{-1} \in M$, so $k \in k_1(M)$. Similarly, $(h, 1) \in L$ and $(1, g) = (h, g)(h, 1)^{-1} \in L$, so $g \in k_2(L)$. Thus, $(k, g) \in k_1(M) \times k_2(L)$, and we conclude that $\ker\theta = k_1(M) \times k_2(L)$. Thus, the first sequence is exact.

Now an element $k \in K$ is in $k_1(M * L)$ if and only if there exists $h \in H$ for which $(k, h) \in M$ and $(h, 1) \in L$. In this case, $h \in p_2(M) \cap k_1(L)$. Therefore, the image of the group $k_1(M * L) \times 1$ by the morphism θ defined above is precisely $\big(p_2(M) \cap$

$k_1(L))/(k_2(M) \cap k_1(L))$. Moreover, its intersection with the kernel $k_1(M) \times k_2(L)$ of θ is $k_1(M) \times 1$. This is sufficient to define the second exact sequence. The third follows similarly.

2. It is obvious that $k_1(M * L) \subseteq p_1(M * L)$ from the definitions. Now if $k \in k_1(M)$, then $(k, 1) \in M$ and $(1, 1) \in L$, so $k \in k_1(M * L)$. Finally, if $k \in p_1(M * L)$, then there exists $h \in H, g \in G$ for which $(k, h) \in M$ and $(h, g) \in L$, so in particular, $k \in p_1(M)$. The second line follows similarly.

□

Remark 2.3.23 (Factorization of Transitive Bisets)**.** If G, H are groups, then by Lemma (2.3.4), any (H, G)-biset is a disjoint union of transitive (H, G)-bisets, and a transitive (H, G)-biset is isomorphic to $(H \times G)/L$ for some $L \leqslant H \times G$ (recall L is the stabilizer of some $u \in U$ in $H \times G$).

Lemma 2.3.24 (Mackey formula for bisets)**.** Let G, H, K be groups. If $L \leqslant H \times G$ and $M \leqslant K \times H$, then there is an isomorphism of (K, G)-bisets

$$\big((K \times H)/M\big) \times_H \big((H \times G)/L\big) \cong \bigsqcup_{h \in [p_2(M)\backslash H/p_1(L)]} (K \times G)/(M * {}^{(h,1)}L),$$

where $[p_2(M)\backslash H/p_1(L)]$ is a set of representatives of double cosets.

Proof. Set $V := (K \times H)/M$ and $U = (H \times G)/L$. We verify that the map:

$$\phi : K\big((k, h)M,_H (h', g)L\big)G \mapsto p_2(M)(h^{-1}h')p_1(L)$$

is a bijection of biset orbits between $K\backslash(V \times_H U)/G \to p_2(M)\backslash H/p_1(L)$.

First to verify the map is well-defined, we must check two things, first that the choice of representative of the $V \times_H U$ term does not matter, and second, that two elements in the same (K, G)-orbit are sent to the same orbit. First note if

$$\big((k_1, h_1)M,_H (h'_1, g_1)L\big) = \big((k_2, h_2)M,_H (h'_2, g_2)L\big),$$

then there exists $h \in H$ for which

$$\begin{aligned}\big((k_1, h_1)M, (h'_1, g_1)L\big) \cdot h &= \big([(k_1, h_1)M] \cdot h, h^{-1} \cdot [(h'_1, g_1)L]\big)\\ &= \big((k_1, h^{-1}h_1)M, (h^{-1}h'_1, g_1)L\big)\\ &= \big((k_2, h_2)M, (h'_2, g_2)L\big)\end{aligned}$$

so $h^{-1}h_1 = h_2$ and $h^{-1}h'_1 = h'_2$, and it is a quick check from there that ϕ is independent of choice of representative in $V \times_H U$. Moreover, it is clear that any two elements in the same orbit are sent to the same orbit, i.e. ϕ is invariant with respect to the left K and right G

actions, since the K and G actions do not affect the H-terms.

To see that the map is a bijection, the inverse map is given by $\phi^{-1} : p_2(M)\backslash H/p_1(L) \to K\backslash(V \times_H U)/G$, given by

$$p_2(M)hp_1(L) \mapsto K\big((1,1)M,_H (h,1)L\big)G.$$

First to check that the map is well-defined we check that if h_1 and h_2 live in the same orbit, then they are sent to the same orbit. If $p_2(M)h_1p_1(L) = p_2(M)h_2p_1(L)$, there exist $h_M \in p_2(M), h_L \in p_1(L)$ such that $h_1 = h_M h_2 h_L$, and $(h_L, g) \in L$ and $(k, h_M) \in M$ for some $k \in K, g \in G$. Then

$$\begin{aligned} p_2(M)h_1p_1(L) \mapsto K\big((1,1)M,_H (h_1,1)L\big)G &= K\big((k^{-1}, h_M^{-1})M,_H (h_1h_L, g)L\big)G \\ &= K\big((k^{-1}, h_M^{-1})M \cdot h_M^{-1},_H h_M(h_1h_L, g)L\big)G \\ &= K\big((k^{-1}, 1)M,_H (h_Mh_1h_L, g)L\big)G \\ &= K\big((1,1)M,_H (h_2,1)L\big)G \leftarrow\!\shortmid p_2(M)h_2p_1(L) \end{aligned}$$

Thus the map is well-defined. Finally, it is straightforward to see that these are indeed inverse maps, so we first conclude that we have an isomorphism of biset orbits.

Now, we have from (2.3.4) a isomorphism of (K, G)-bisets:

$$\big((K \times H)/M\big) \times_H \big((H \times G)/L\big) \cong \bigsqcup_{x \in [K\backslash(V \times_H U)/G]} (K \times G)/(K, G)_x,$$

however by the isomorphism we may rewrite this as:

$$\big((K \times H)/M\big) \times_H \big((H \times G)/L\big) \cong \bigsqcup_{u \in [p_2(M)\backslash H/p_1(L)]} (K \times G)/(K, G)_{\phi^{-1}(u)}.$$

Finally, we must compute $(K, G)_{\phi^{-1}(u)}$, i.e. the stabilizer of $\big((1,1)M,_H (h,1)L\big)$ in $K \times G$. It is clear that M is the stabilizer of $(1,1)M$. Moreover, it straightforward to verify that ${}^{(h,1)}L = (h,1)L(h^{-1},1)$ is the stabilizer of $(h,1)L$, so finally by (2.3.20) we conclude $(K, G)_{\phi^{-1}(u)} = M * {}^{(h,1)}L$ and the result follows.

The book calls this verification "easy to check." Ha. Ha. □

Lemma 2.3.25 (Goursat Lemma)**.** Let G, H be groups.

1. If (D, C) is a section of H and (B, A) is a section of G such that there exists a group isomorphism $f : B/A \to D/C$ then

$$L_{(D,C),f,(B,A)} = \{(h, g) \in H \times G : h \in D, g \in B, hC = f(gA)\}$$

is a subgroup of $H \times G$.

2. Conversely, if L is a subgroup of $H \times G$, then there exists a unique section (D, C) of H, a unique section (B, A) of G, and a unique group isomorphism $f : B/A \to D/C$ such that $L = L_{(D,C),f,(B,A)}$.

Proof. 1. This is an easy verification, all that is required to check is that the subgroup is indeed a subgroup.

2. We assert that the choices:

$$D = p_1(L), \qquad B = p_2(L),$$

$$C = k_1(L), \qquad A = k_2(L),$$

and $f : B/A \to D/C$ determined by $f(bA) = dC$ when $(d, b) \in L$ satisfy $L = L_{(D,C),f,(B,A)}$ and are unique. First, it follows that these choices form sections of H and G, with isomorphic associated subquotients from (2.3.18) (recall that $u = (1,1)L$), and moreover, the map f corresponds exactly to the map $\overline{c_u}$ in (2.3.18.3), so it is an isomorphism. They correspond since $(d, b) \in L$ if and only if $d \cdot (1,1)L = (1,1)L \cdot b$. Now it is clear from definitions that $L_{(D,C),f,(B,A)} = L$.

It follows that this choice is unique, since $p_1(L_{(D,C),f,(B,A)}) = D$, $p_2(L_{(D,C),f,(B,A)}) = B$, et cetera. So if we have $L = L_{(D,C),f,(B,A)} = L_{(D',C'),f',(B',A')}$, it immediately follows that $A = A'$, $B = B'$, and so on.

□

The next result in some sense allows us to consider the 5 elementary bisets as "essential," as in, every biset $(H \times G)/L$ can be decomposed as a composition of those 5 bisets.

Theorem 2.3.26. Let G and H be groups. If $L \leqslant H \times G$, let (D, C) and (B, A) be sections of H, G respectively and let f be the group isomorphism $B/A \xrightarrow{\cong} D/C$ such that $L = L_{(D,C),f,(B,A)}$. Then there is an isomorphism of (H, G)-bisets:

$$(H \times G)/L \cong \mathrm{Ind}_D^H \times_D \mathrm{Inf}_{D/C}^D \times_{D/C} \mathrm{Iso}(f) \times_{B/A} \mathrm{Def}_{B/A}^B \times_B \mathrm{Res}_B^G$$

Proof. Define $\Lambda = (H \times G)/L$ and let Γ denote the right-hand side of the proposed isomorphism. Define the map $\varphi : \Lambda \to \Gamma$ by

$$\varphi : (h, g)L \mapsto (h,_D C,_{D/C} C,_{B/A} A,_B g^{-1})$$

(note that the middle terms are all identity cosets) and $\psi : \Gamma \to \Lambda$ by

$$\psi : (h,_D, dC,_{D/C}, d'C, B/A, bA,_B g) \mapsto (hdd', g^{-1}b^{-1})L,$$

for $h \in H, d, d' \in D, b \in B, g \in G$. First we verify that these maps are well defined: suppose $(h,g) \in H \times G$ and $(d,b) \in L$. Note f satisfies $f(bA) = dC$ by the previous lemma. Thus

$$\begin{aligned}\varphi\big((hd, gb)L\big) &= (hd,_D C,_{D/C} C,_{B/A} A,_B b^{-1}g^{-1}) \\ &= (h,_D dC,_{D/C} C,_{B/A} Ab^{-1},_B g^{-1}) \\ &= (h,_D C(dC),_{D/C}, C,_{B/A} (b^{-1}A)A,_B g^{-1}) \\ &= (h,_D C,_{D/C} (dC)C(bA)^{-1},_{B/A} A,_B g^{-1})\end{aligned}$$

However, since $f(bA) = dC$, we see that $(dC)C(bA)^{-1} = C \in \mathrm{Iso}(f)$, and hence, φ is indeed well-defined. Similarly, we check ψ is well-defined. Let $x \in D, yC \in D/C, zA \in B/A, y' \in f(zA), t \in B$. We verify that the image of ψ of the element

$$E = (hx,_D x^{-1}dCy,_{D/C} y^{-1}d'Cy',_{B/A} z^{-1}bAt, t^{-1}g) \in \Gamma$$

should be equal to $\psi(h,_D, dC,_{D/C}, d'C,_{B/A}, bA,_B g)$. We compute:

$$\begin{aligned}\psi(E) &= (hxx^{-1}dyy^{-1}d'y', g^{-1}tt^{-1}b^{-1}z)L \\ &= (hdd'y', g^{-1}b^{-1}z)L \\ &= (hdd', g^{-1}b^{-1})L\end{aligned}$$

The final line comes from the fact that $(y', z) \in L$, since $f(zA) = y'C$, implying $(y', z) \in L_{(D,C),f,(B,A)} = L$. Thus both maps are indeed well-defined. It is straightforward to check that these maps are (H,G)-equivariant, so they are indeed maps of bisets. Finally, we check that they are inverse. It is obvious that $\psi \circ \varphi = \mathrm{Id}_\Lambda$. We compute:

$$\begin{aligned}\varphi \circ \psi\big((h,_D, dC,_{D/C} d'C,_{B/A} bA,_B g)\big) = \varphi\big((hdd', g^{-1}b^{-1})L\big) &= (hdd',_D C,_{D/C} C,_{B/A} A,_B bg) \\ &= (h,_D dd'C,_{D/C} C,_{B/A} A(bA),_B g) \\ &= (h,_D (dC)(d'C),_{D/C} C,_{B/A} bA,_B g) \\ &= (h,_D dC,_{D/C} d'C,_{B/A} bA,_B g)\end{aligned}$$

as desired. □

In short, *any* transitive (H,G)-biset can be uniquely realized as the composition of these 5 fundamental bisets.

Day 3

2.5 The Burnside Ring (cont.)

First recall we have a construction: if G is a group and X is a G-set, we construct the (G,G)-biset $\tilde{X}$ by setting it to be the set $G \times X$ with biset structure given by

$$a \cdot (g, x) \cdot b = (agb, b^{-1}x)$$

Let us recall the maps given in (2.5.10). Let G, H be groups and U a (H,G)-biset. If X is a H-set, then:

$$\alpha_{U,X} : \tilde{X} \times_H U \to U \times_G (\widetilde{U^{op} \times_H X})$$
$$\big((h,x),_H u\big) \to \Big(hu,_G \big(1, (u,_H x)\big)\Big)$$

is well-defined. Additionally, if Y is a G-set then

$$\beta_{U,Y} : \widetilde{U \times_G Y} \to U \times_G \tilde{Y} \times_G U^{op}$$
$$\big(h, (u,_G y)\big) \mapsto \big(hu,_G (1,y),_G u\big)$$

is well defined. If U is left-free then α, β are both injective, and if U is left-transitive then α, β are both surjective, regardless of X and Y.

Corollary 2.5.1. Let G be a group and let $H \leqslant G$.

1. Let X be a G-set. Then there is an isomorphism of (G,H)-bisets

   $$\tilde{X} \times_G \mathrm{Ind}_H^G \cong \mathrm{Ind}_H^G \times_H \widetilde{\mathrm{Res}_H^G X}$$

   and an isomorphism of (H,G)-bisets

   $$\mathrm{Res}_H^G \times_G \tilde{X} \cong \widetilde{\mathrm{Res}_H^G X} \times_H \mathrm{Res}_H^G$$

2. Let Y be an H-set. Then there is an isomorphism of (G,G)-bisets

   $$\mathrm{Ind}_H^G \times_H \tilde{Y} \times_H \mathrm{Res}_H^G \cong \widetilde{\mathrm{Ind}_H^G Y}$$

Proof. Note that $\mathrm{Res}_H^G X$ refers to the left G-set $\mathrm{Res}_H^G \times_H X$ and $\mathrm{Ind}_H^G Y$ refers to the left G-set $\mathrm{Ind}_H^G \times_H Y$.
The first isomorphism in part 1 and the isomorphism in part 2 follow from the previous proposition, by switching G and H and letting U be the (G,H)-biset Ind_H^G. Recall Ind_H^G is the set G itself. It is obvious that Ind_H^G is left-free and left-transitive, so α and β are isomorphisms. Moreover, it is clear that $(\mathrm{Ind}_H^G)^{op} \cong \mathrm{Res}_H^G$. Now, the first isomorphism in part 1 follows via the map α and part 2 follows from the map β.

Now, note that it is clear that given bisets A, B, $A \cong B$ if and only if $A^{op} \cong B^{op}$. Using this fact, the second isomorphism in part 1 follows from the first isomorphism, the fact that $(V \times_H U)^{op} \cong U^{op} \times_H V^{op}$ for compatible bisets U, V, and the observation that $\tilde{X} \cong \tilde{X}^{op}$. Most of these facts are clear or have been proven before, but the last isomorphism needs verifying. We construct a biset map $\phi : \tilde{X} \to \tilde{X}^{op}$ by $(g, x) \mapsto (g^{-1}, gx)$. We verify that this map is (G, G)-equivariant:

$$\begin{aligned} a \cdot \phi(g, x) \cdot b &= a \cdot (g^{-1}, gx) \cdot b \\ &= (b^{-1}g^{-1}a^{-1}, agx) \\ &= \big((agb)^{-1}, agx\big) \\ &= \big((agb)^{-1}, (agb)b^{-1}g\big) \\ &= \phi(agb, b^{-1}g) \\ &= \phi\big(a \cdot (g, x) \cdot b\big) \end{aligned}$$

This map has inverse $\phi^{-1} : \tilde{X}^{op} \to \tilde{X}$ also given by $(g, x) \mapsto (g^{-1}, gx)$. Equivariance follows similarly as before, and it is straightforward to compute that these maps are inverse. Thus the bisets are isomorphic, as desired. □

Corollary 2.5.2. Let G be a group.

1. Let N be a normal subgroup of G, and X be a (G/N)-set. Then there is an isomorphism of $(G/N, G)$-bisets
$$\tilde{X} \times_{G/N} \operatorname{Def}^G_{G/N} \cong \operatorname{Def}^G_{G/N} \times_G \widetilde{\operatorname{Inf}^G_{G/N} X},$$
and an isomorphism of $(G, G/N)$-bisets
$$\operatorname{Inf}^G_{G/N} \times_{G/N} \tilde{X} \cong \widetilde{\operatorname{Inf}^G_{G/N} X} \times_G \operatorname{Inf}^G_{G/N}.$$

2. Let N be a normal subgroup of G and Y be a G-set. Then there is an isomorphism of $(G/N, G/N)$-bisets
$$\operatorname{Def}^G_{G/N} \times_G \tilde{Y} \times_G \operatorname{Inf}^G_{G/N} \cong \widetilde{\operatorname{Def}^G_{G/N} Y}$$

Proof. The proof follows the same as the previous one, now using the assignment $H = G/N$ and $U = \operatorname{Def}^G_{G/N}$. □

We conclude the section by looking at a proposition which considers the case when G and H have coprime order. We first revisit a necessary fact:

Lemma 2.5.3. Let G be a group and $[s_G]$ a set of representatives of conjugacy classes of subgroups of G and H. Then $B(G)$ is a free abelian group with basis $\mathcal{B} = \{[G/S] : S \in [s_G]\}$.

Proof. First, recall from (2.2.2) that every G-set X can be written (up to isomorphism) as

$$X = \bigsqcup_{K \in [s_G]} a_K(X) G/K$$

where $a_K(X) \in \mathbb{N}$. Therefore, $\mathcal{B}$ is a spanning set of $B(G)$. It remains to show $\mathcal{B}$ is linearly independent. Let us suppose we have a relation

$$0 = \sum_{S \in [s_G]} a_S [G/S]$$

with each $a_S \in \mathbb{Z}$. Denote $[s_G]^+$ to be the subset of $S \in [s_G]$ with $a_S > 0$ and $[s_G]^-$ to be the subset with $a_S < 0$. If we can show these sets are empty, we are done.
We can rearrange terms so all coefficients are positive as follows:

$$\sum_{S \in [s_G]^+} a_S [G/S] = \sum_{T \in [s_G]^-} (-a_T)[G/T]$$

Now each sum is the image of some G-set in $B(G)$ as follows:

$$X = \bigsqcup_{S \in [s_G]^+} a_S G/S, \qquad Y = \bigsqcup_{T \in [s_G]^-} (-a_T) G/T.$$

Since their images in $B(G)$ are equal, $X \cong Y$. Now, suppose for contradiction that $[s_G]^+$ or $[s_G]^-$ is nonempty. Then, consider the poset of $[s_G]^+ \cup [s_G]^-$ ordered by inclusion up to conjugation. Since these sets are finite, some maximal element H must exist. Suppose without loss of generality $H \in [s_G]^+$, then $|X^H| = a_S[G : H]$ by maximality. However, by (2.4.5), then $|Y^H| = a_S[G : H] > 0$. Since $H \in [s_G]^+$, $H \notin [s_G]^-$, but the only sets in $\mathcal{B}$ fixed by H are $[G/H']$ with $H \leqslant H'$, contradicting maximality. Thus $[s_G]^+ = [s_G]^- = \varnothing$, as desired. ☐

Proposition 2.5.4. Let G, H be finite groups.

1. If X is a G-set and Y is an H-set, then $X \times Y$ is a $(G \times H)$-set with action defined componentwise. The correspondence $(X, Y) \mapsto X \times Y$ induces a bilinear map $B(G) \times B(H) \to B(G \times H)$ and hence a homomorphism

   $$\pi : B(G) \otimes_{\mathbb{Z}} B(H) \to B(G \times H)$$

   which is an injective ring homomorphism preserving identity elements. If G and H have coprime order, this map is an isomorphism.

2. If U is a (G, G)-biset and V is an (H, H)-biset, then $U \times V$ is a $(G \times H, G \times H)$-biset for the structure given again by componentwise multiplication, i.e.

   $$(g, h) \cdot (u, v) \cdot (g', h') = (gug', huh').$$

The correspondence $(U, V) \mapsto U \times V$ induces a bilinear map $B(G, G) \times B(H, H) \to B(G \times H, G \times H)$, hence a linear map

$$\pi_2 : B(G, G) \otimes_{\mathbb{Z}} B(H, H) \to B(G \times H, G \times H)$$

which is an injective ring homomorphism preserving identity elements. If G and H have coprime orders, then this map is an isomorphism.

Proof. 1. The correspondence $\tilde{\pi}(X, Y) \mapsto X \times Y$ induces an obvious map $B(G) \times B(H) \to B(G \times H)$ which is bilinear, since $(X_1 \sqcup X_2) \times Y \cong (X_1 \times Y) \sqcup (X_2 \times Y)$, and similarly in the second argument. Thus by the universal property of tensor products, a linear map $\pi : B(G) \otimes_{\mathbb{Z}} B(H) \to B(G \times H)$ is induced. We verify that π respects the multiplicative structure: we must show that if X, X' are G-sets and Y, Y' are H-sets, then $\tilde{\pi}(X, Y) \times \tilde{\pi}(X', Y') \cong \tilde{\pi}\big((X \times X'), (Y \times Y')\big)$. This follows since

$$(X \times Y) \times (X' \times Y') \cong (X \times X') \times (Y \times Y')$$

is an obvious isomorphism of $(G \times H)$-sets, so their images in the induced map are equal. Hence π is a ring homomorphism. Finally, π is unital, since if X is a G-set of cardinality 1 and Y is a H-set of cardinality 1, $X \times Y$ is a $(G \times H)$-set of cardinality 1.

Let $[s_G]$ and $[s_H]$ denote sets of representatives of conjugacy classes of subgroups of G and H respectively. Then $B(G)$ is free abelian with basis $\{[G/S] : S \in [s_G]\}$, and $B(H)$ is free abelian with basis $\{[H/T] : T \in [s_H]\}$. Then, $B(G) \otimes_{\mathbb{Z}} B(H)$ is free abelian with basis $\mathcal{B} = \{[G/S] \otimes [H/T] : (S, T) \in [s_G] \times [s_H]\}$.

It is clear that $\pi([G/S] \otimes [H/T]) = [(G \times H)/(S \times T)]$. Moreover, the subgroups $S \times T$ lie in different conjugacy classes of subgroups of $G \times H$, where $(S, T) \in [s_G] \times [s_H]$, so $\pi(\mathcal{B})$ is a subset of a $\mathbb{Z}$-basis of $B(G \times H)$. Thus π is injective.

Recall if G, H are groups, and $L \leqslant G \times H$, we defined:

$$\begin{aligned}
p_1(L) &= \{h \in H : \exists g \in G,\ (h, g) \in L\} \\
p_2(L) &= \{g \in G : \exists g \in G,\ (h, g) \in L\} \\
k_1(L) &= \{h \in H : (h, 1) \in L\} \\
k_2(L) &= \{g \in G : (1, g) \in L\} \\
q(L) &= L/(k_1(L) \times k_2(L))
\end{aligned}$$

Now if G, H have coprime orders and $L \leqslant G \times H$, then $q(L) = 1$, since $q(L) \cong p_1(L)/k_1(L) \cong p_2(L)/k_2(L)$. From this it is clear to see that $L = k_1(L) \times k_2(L) = p_1(L) \times p_2(L)$. Hence, if $L \leqslant G \times H$, then $L = S \times T$ for some subgroups $S \leqslant G, T \leqslant H$. Since a $\mathbb{Z}$-basis of $B(G \times H)$ is $\mathcal{B}' = \{[G \times H/L] : L \in [s_{G \times H}]\}$, it follows that $\pi(\mathcal{B}) = \mathcal{B}'$ and hence π is surjective, and therefore an isomorphism.

2. Set $G_2 = G \times G^{op}$ and $H_2 = H \times H^{op}$. Part 1 gives a linear map:

$$\pi : B(G_2) \otimes_{\mathbb{Z}} B(H_2) \to B(G_2 \times H_2),$$

and $G_2 \times H_2 \cong (G \times H)_2 = (G \times H) \times (G \times H)^{op}$. Composing this gives a map $B(G_2) \otimes_{\mathbb{Z}} B(H_2) \to B\big((G \times H)_2\big)$. It follows from the definition of bisets that we may identify this map with the map

$$\pi_2 : B(G,G) \otimes_{\mathbb{Z}} B(H,H) \to B(G \times H, G \times H)$$

since $B(G_2) = B(G,G)$ and so on. It follows from part 1 that π_2 is injective, and an isomorphism if G and H have coprime orders.

It remains to verify that π_2 is a ring homomorphism under biset multiplication. We wish to show that if U, U' are (G,G)-bisets and V, V' are (H,H)-bisets, then $\pi_2(U \times_G U', V \times_H V') = \pi(U,V) \cdot \pi(U',V')$. However, there is an isomorphism of $(G \times H, G \times H)$-bisets given by:

$$(U \times_G U') \times (V \times_H V') \cong (U \times V) \times_{G \times H} (U' \times V')$$
$$\big((u,_G u'), (v,_H v')\big) \mapsto \big((u,v),_{G \times H} (u',v')\big)$$

and multiplicity of π_2 follows. Finally it is obvious that the map sends identity bisets to identity bisets.

□

3.1 The Biset Category of Finite Groups

Definition 3.1.1. The biset category $\mathcal{C}$ of finite groups is the category defined as follows:

- The objects are finite groups.
- If G and H are finite groups, $\mathrm{Hom}_{\mathcal{C}}(G,H) = B(H,G)$.
- If G, H, K are finite groups, and $u \in \mathrm{Hom}_{\mathcal{C}}(G,H)$ and $v \in \mathrm{Hom}_{\mathcal{C}}(H,K)$, then $v \circ u := v \times_H u$.
- For any finite group G, the identity morphism of G in $\mathcal{C}$ is $[\mathrm{Id}_G]$.

Remark 3.1.2. It follows that $\mathcal{C}$ is preadditive (in the sense of MacLane) - the morphism sets are abelian groups and composition is bilinear.
If G and H are finite groups, then any morphism from G to H in $\mathcal{C}$ is a linear combination with integral coefficients of morphism of the form $[(H \times G)/L]$ where $L \leqslant H \times G$. From (2.3.26) any such morphism factors as follows:

$$G \xrightarrow{\mathrm{Res}^G_B} B \xrightarrow{\mathrm{Def}^B_{B/A}} B/A \xrightarrow{\mathrm{Iso}(f)} D/C \xrightarrow{\mathrm{Inf}^D_{D/C}} D \xrightarrow{\mathrm{Ind}^H_D} H$$

In other words $\mathcal{C}$ is generated as a preadditive category by the five types of morphisms above, associated to elementary bisets.
Now let ${}^*\mathcal{C}$ be the preadditive category whose objects are finite groups and morphisms are $\mathbb{Z}$-generated by elementary morphisms,

- ${}^*\operatorname{Res}_H^G : G \to H$
- ${}^*\operatorname{Ind}_H^G : H \to G$
- ${}^*\operatorname{Inf}_{G/N}^G : G/N \to G$
- ${}^*\operatorname{Def}_{G/N}^G : G \to G/N$
- ${}^*\operatorname{Iso}(\phi) : G \to G'$ for $\phi : G \to G'$

These morphisms are subject to a list of relations given in (1.1.3) of the book. Some examples are relations dictating composition (i.e. $\operatorname{Res}_H^G \circ \operatorname{Res}_K^H = \operatorname{Res}_K^G$), identity morphisms (i.e. $\operatorname{Res}_G^G = \operatorname{Id}_G$), and commutation (i.e. the Mackey formula).
One may verify that the correspondence $\Theta : {}^*\mathcal{C} \to \mathcal{C}$ given by sending each group to itself and removing *'s on the elementary morphisms is a functor. Conversely, there is a unique morphism $\Psi : \mathcal{C} \to {}^*\mathcal{C}$ which is the identity on objects, and sends a morphism $G \to H$ defined by a transitive biset $(H \times G)/L$ to the morphism

$${}^*\operatorname{Ind}_D^H \circ {}^*\operatorname{Inf}_{D/C}^D \circ {}^*\operatorname{Iso}(\phi) \circ {}^*\operatorname{Def}_{B/A}^B \circ {}^*\operatorname{Res}_B^G$$

where $D = p_1(L), C = k_1(L), B = p_2(L), A = k_2(L)$, and $\phi : B/A \to D/C$ is the canonical isomorphism from before. One may show this morphism is unique up to conjugation by L so Ψ is well defined. It is (according to the book) a tedious but straightforward task to show that Ψ is a functor, and equivalent to checking that any composition of elementary morphisms in ${}^*\mathcal{C}$ is a sum of morphisms as above.
Then it is clear that Θ and Ψ are mutual inverse equivalence of categories. In other words, the elementary morphisms along with the relations presented in 1.1.3 form **a presentation of the biset category** $\mathcal{C}$.

Remark 3.1.3. Lemma (2.4.11) shows that there is a functor from the biset category to the opposite category which maps any object to itself and any morphism $u \in \operatorname{Hom}_{\mathcal{C}}(G, H) = B(H, G)$ to $u^{op} \in B(G, H) = \operatorname{Hom}_{\mathcal{C}^{op}}(G, H)$. It is obviously an equivalence of categories (in fact, an isomorphism).

It is natural to consider other coefficient rings instead of integers:

Definition 3.1.4. Let R be a commutative ring with identity. The category $R\mathcal{C}$ is defined as follows:

- The objects of $R\mathcal{C}$ are finite groups.

- If G and H are finite groups, then $\mathrm{Hom}_{R\mathcal{C}}(G,H) = R \otimes_{\mathbb{Z}} B(H,G)$

- The composition of morphisms in $R\mathcal{C}$ is the R-linear extension of the composition in $\mathcal{C}$.

- For any finite group G, the identity morphism of G in $R\mathcal{C}$ is equal to $R \otimes_{\mathbb{Z}} \mathrm{Id}_G$(????)

This category is a R-linear category, i.e. the set of morphisms in $R\mathcal{C}$ are R-modules and the composition in $R\mathcal{C}$ is R-bilinear.