Endotrivial Complexes and Tensor Induction

Sam Miller
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Contents:

Big question: let k be a field, G a finite group.

We want to understand the module category

${}_{kG}\text{mod}$:
- Objects: finitely generated kG-modules (finite k-dimension)
- Morphisms: kG-module homomorphisms

(an abelian category)

(more generally, one can decompose kG into its Blocks,

$$kG \cong B_1 \times B_2 \times \cdots \times B_\ell \quad \text{as } k\text{-algebras}$$

and block theorists study ${}_{B}\text{mod}$, a subcategory of ${}_{kG}\text{mod}$)

A subquestion: When are two module categories of group/block algebras equivalent?

(a) (The obvious answer) When there is an equivalence of categories

$$ {}_{kG}\text{mod} \underset{F_2}{\overset{F_1}{\rightleftarrows}} {}_{kH}\text{mod} $$

i.e. functors that are "inverse," up to isomorphisms.

By a theorem of Morita, such equivalences present nicely!

Thm (Morita) Let A, B be k-algebras. TFAE

(a) ${}_A\mathrm{Mod} \cong {}_B\mathrm{Mod}$ (capital M indicates non-finitely generated)

(b) ${}_A\mathrm{mod} \cong {}_B\mathrm{mod}$

(c) There is an A,B-bimodule M and a B,A-bimodule N such that $M \otimes_B N \cong A$ and $N \otimes_A M \cong B$, and such that M, N are finitely generated projective as left or right modules.

Then the equivalences are induced by

$M \otimes_B - : {}_B\mathrm{Mod} \to {}_A\mathrm{Mod}$

$N \otimes_A - : {}_A\mathrm{Mod} \to {}_B\mathrm{Mod}$

and further,

$N \cong \mathrm{Hom}_A(M, A) \cong \mathrm{Hom}_{B^{op}}(M, B)$

$M \cong \mathrm{Hom}_B(N, B) \cong \mathrm{Hom}_{A^{op}}(N, A)$

If A, B are group/block algebras then moreover,

$N \cong M^* = \mathrm{Hom}_k(M, k)$

(b) (The character-theoretic answer)

When there is a "perfect" isometry between virtual character rings.

Recall from John's talk: over a field K with characteristic 0, the set of class functions

$$CF(G,K) = \{ f : G \to K \mid f \text{ constant on conjugacy classes of } G \}$$

is a K-vector space, and has the Schur inner product

$$(f_1, f_2) = \frac{1}{|G|} \sum_{g \in G} f_1(g) f_2(g^{-1})$$

Then the set of irreducible characters of G forms an orthonormal basis of $CF(G,K)$.

Def: (sketch)

A perfect isometry from kG to kH is an K-linear iso:

$$I : CF(G,K) \to CF(H,K)$$

which

(1) is an isometry (distance preserving)

(2) sends irreps of G to $(\pm)$irreps of H

(3) satisfies a few other "perfect" conditions, relating the classical character theory with "modular" representations. (See John's talk)

(2) implies that these maps restrict to isomorphisms of virtual character rings $R_K(G) \cong R_K(H)$

$$R_K(G) = \{ \mathbb{Z}\text{-linear combinations of irreducible characters} \}$$

Side note:

Perfect isometries exist within the context of p-modular systems, a trio of rings $(K, \mathcal{O}, k)$

- K - field of characteristic 0
- k - field of characteristic p
- $\mathcal{O}$ - a ring such that $\mathcal{O}/\pi = k$, $K = \mathrm{Frac}\,\mathcal{O}$

e.g. $(K = \mathbb{Q}_p,\ \mathcal{O} = \mathbb{Z}_p,\ k = \{\text{finite field}\})$

This allows us to relate "modular" representation theory to classical representation theory

It can be shown that "perfect" $(\mathcal{O}G, \mathcal{O}H)$-bimodules, bimodules which are projective as left or right modules, induce perfect isometries!

(C) (using categorical constructions)

Given a module category ${}_{kG}\text{mod}$ we can form new categories

- $K^b({}_{kG}\text{mod})$ - the (bounded) homotopy category of (f.g.) kG-modules
- $D^b({}_{kG}\text{mod})$ - the (bounded) derived category of (f.g) kG-modules

$K^b({}_{kG}\text{mod})$
- objects are bounded chain complexes of f.d. kG-modules

$$\left(0 \longrightarrow M_n \xrightarrow{d_n} M_{n-1} \xrightarrow{d_{n-1}} M_{n-2} \longrightarrow \cdots \xrightarrow{d_1} M_0 \longrightarrow 0 \cdots \right) \qquad d \circ d = 0$$

- morphisms are homotopy classes of chain complex morphisms

It can be the case that while there is not a Morita equivalence

$${}_{kG}\text{mod} \not\cong {}_{kH}\text{mod}$$

we do have equivalences on the homotopy/derived level

$$K^b({}_{kG}\text{mod}) \cong K^b({}_{kH}\text{mod})$$

or

$$D^b({}_{kG}\text{mod}) \cong D^b({}_{kH}\text{mod})$$

Jeremy Rickard proved that such equivalences are "equivalent".

<u>Thm</u> (Rickard) The following are equivalent:

(1) $D^b({}_{kG}\text{mod}) \cong D^b({}_{kH}\text{mod})$

(2) There is a bounded complex of perfect (kG, kH)-bimodules X such that

$$X \otimes_{kH} X^* \simeq kG \qquad \text{homotopy equivalent, i.e. isomorphic in homotopy cat}$$
$$X^* \otimes_{kG} X \simeq kH \qquad \text{regarded as chain complexes in deg } 0$$

which induces an equivalence via tensoring,

$$K^b({}_{kG}\text{mod}) \cong K^b({}_{kH}\text{mod})$$

So,
- Morita equivalence ⟿ "invertible" bimodule
- Derived/Homotopy equivalence ⟿ "invertible" chain complex of bimodules

How do these equivalences compare?

(a) Morita eq. $\Longrightarrow$ (c) Derived/Homotopy eq. $\Longrightarrow$ (b) Perfect Isometry

What about: (recalling John's talk) isotypies? (b2)

(a "bouquet" of compatible perfect isometries, describing local behavior)

In general, an arbitrary Morita or derived equivalence does not induce an isotypy.

$$\left[\begin{tikzcd}
{[??]} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & {[??]} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{Isotypy} \arrow[d, Rightarrow] \\
\text{Morita eq.} \arrow[r, Rightarrow] & \text{Derived/Homotopy eq.} \arrow[r, Rightarrow] & \text{Perfect Isometry}
\end{tikzcd}\right]$$

Rickard solved this question in 1994! (the paper was published one day before my birth!)

<u>Def:</u> Let $C_\bullet$ be a complex of perfect (kG, kH)-bimodules, which, when restricted to any p-subgroup of $G \times H$, is a permutation module.

If

homotopy equivalent, i.e. isomorphic in $K^b(-\text{''}-)$

$$C \otimes_{kH} C^* \simeq kG$$
$$C^* \otimes_{kG} C \simeq kH$$

regarded as complexes in degree 0

we say C is a <u>"splendid" Rickard complex.</u>

(Such modules are <u>p-permutation modules</u>)

Note that a splendid Rickard complex is a special homotopy or derived equivalence.

<u>Thm:</u> (Rickard)

Splendid Rickard complexes induce isotypies!

Here we have a commuting diagram of equivalences:

$$\begin{tikzcd}
\text{"Splendid" Morita eq.} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{"Splendid" derived eq.} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{Isotypy} \arrow[d, Rightarrow] \\
\text{Morita eq.} \arrow[r, Rightarrow] & \text{Derived/Homotopy eq.} \arrow[r, Rightarrow] & \text{Perfect Isometry}
\end{tikzcd}$$

"Splendid" stands for: Split endomorphism two-sided tilting complex induced from diagonal subgroups,

which refers to chain complexes - however we tend to use "splendid" to refer to any equivalence with p-permutation modules.

(d) Comparing Grothendieck groups

Given a category of R-modules, we define its Grothendieck group (with respect to direct sums) is the abelian group formed by taking $\mathbb{Z}$-linear sums of R-modules, with addition given by

$$[M] + [N] = [M \oplus N]$$

Each element of the group looks like:

$$X = [M] - [N]$$

for some (nonunique) R-modules M, N.

We call these virtual modules.

Idea: • Another equivalence ${}_{kG}\text{mod}$ "$\cong$" ${}_{kH}\text{mod}$ could be defined by isomorphisms on their Grothendieck groups!

- Like how Morita/Derived equivalences arise from tensoring by "invertible" bimodules, we can induce virtual equivalences by tensoring by invertible virtual bimodules.

Boltje and Xu proved that splendid virtual equivalences lay between splendid derived equivalences and isotypies!

- Denote by $T(kG)$ the Grothendieck group of p-permutation kG-modules. It is a ring via the tensor product.
- Denote by $T(kG, kH)$ the Grothendieck group of p-permutation (kG, kH)-bimodules.

If $X \in T(kG,kH)$, then $X^* \in T(kH,kG)$ and

$$X \otimes_{kH} X^* \in T(kG,kG) \quad , \quad X^* \otimes_{kG} X \in T(kH,kH)$$

<u>Def</u>: A p-permutation equivalence between kG and kH is a perfect virtual bimodule $X \in T(kG,kH)$ which satisfies

$$X \otimes_{kH} X^* = [kG] \qquad\qquad X^* \otimes_{kG} X = [kH]$$

This induces an isomorphism $T(kH) \cong T(kG)$.

<u>Thm</u> (Boltje, Xu)

Splendid derived eq. $\Rightarrow$ p-permutation eq. $\Rightarrow$ isotypy

To sum up, our picture is:

My focus

$$\begin{tikzcd}
 & \text{homotopy} \arrow[d, Leftrightarrow] & & \\
\text{Splendid Morita equivalence} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{Splendid derived equivalence} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{p-permutation equivalence} \arrow[r, Rightarrow] & \text{isotypy} \arrow[d, Rightarrow] \\
\text{Morita equivalence} \arrow[r, Rightarrow] & \text{derived equivalence} \arrow[rr, Rightarrow] & & \text{perfect isometry} \\
 & \text{homotopy} \arrow[u, Leftrightarrow] & &
\end{tikzcd}$$

<u>My question</u>: When does a p-permutation equivalence lift to a splendid Rickard complex?

There are more notions as well!

- Source algebra equivalence
- Equivalent fusion systems
- "Stable" category equivalences

(2) Endotrivial complexes

We start by looking at p-permutation auto-equivalences for kG.
One source of them is the unit group of the Burnside Ring.

<u>Def</u>: For a finite group G, denote by $B(G)$ the Grothendieck group of G-sets with addition corresponding to disjoint unions.

(So any $X \in B(G)$ is written $X = [M] - [N]$ for two G-sets M, N)

$B(G)$ is a ring by taking direct products.

<u>Recall</u>: Given groups $H \leq G$, and a kH-module M, there is a kG-module

$$\mathrm{Ind}_H^G M := kG \otimes_{kH} M$$

Similarly one can induce H-sets to G-sets via, for $K \leq H$,

$$H/K \longmapsto G/K$$

<u>Prop</u>: We have a commutative diagram of ring morphisms

$$\begin{tikzcd}
B(G) \arrow[r, hook, "\mathrm{Ind}_{\Delta G}^{G\times G}"] \arrow[d, "k[-]"] & B(G,G) \arrow[d, "k[-]"] \\
T(kG) \arrow[r, hook, "\mathrm{Ind}_{\Delta G}^{G\times G}"] & T(kG,kG)
\end{tikzcd}$$

$[G/H] \longmapsto [G\times G/\Delta H]$

$k[G/H] \longmapsto kG \otimes_{kH} kG$

← Biset Burnside ring

← k-linearization

$k\Delta G$
⇝
kG-module

Inducing a kG-module to a $k[G\times G]$-module, then identifying
$k[G\times G]$-module
⇕
(kG,kG)-bimodule

which restricts to group homomorphisms:

(★)

$$\begin{tikzcd}
B(G)^\times \arrow[r, hook, "\mathrm{Ind}_{\Delta G}^{G\times G}"] \arrow[d, "k[-]"] & O(B(G,G)) \arrow[d, "k[-]"] \\
O(T(kG)) \arrow[r, hook, "\mathrm{Ind}_{\Delta G}^{G\times G}"] & \underbrace{O(T^\Delta(kG,kG))}_{\text{p-permutation autoequivalences}}
\end{tikzcd}$$

O - orthogonal unit group w.r.t. taking duals

Δ - perfect bimodules
⇕
"twisted diagonal vertices"

Big Question

For any unit in $B(G)^\times$, can we find a splendid Rickard complex Γ_u which is compatible with this setup?

$$\begin{tikzcd}
u \in B(G)^\times \arrow[r, "\mathrm{Ind}_{\Delta G}^{G\times G}"] \arrow[d, "k[-]"'] \arrow[ddr, dashed, bend right] & O(B(G,G)) \arrow[d, "k[-]"] \\
O(T(kG)) \arrow[r, "\mathrm{Ind}_{\Delta G}^{G\times G}"'] & O(T^\Delta(kG,kG)) \\
& \Gamma_u \in Spl(kG,kG) \arrow[u, "\Lambda(-)"']
\end{tikzcd}$$

$\leftarrow$ alternating sum

$\leftarrow$ set of splendid Rickard complexes $kG \to kG$

- Old strategy: construct potential splendid Rickard complexes, verify they satisfy the conditions.

  Cons: Very difficult to work with complexes of bimodules, functorial constructions such as inflation very non-obvious

- New strategy: define an analogue of splendid Rickard complexes for left kG-modules.

Def: An endotrivial complex C is a bounded chain complex of p-permutation kG-modules which satisfies:

$$C \otimes_k C^* \cong C^* \otimes_k C \simeq k$$

Why "endotrivial?"

There is a canonical isomorphism $C \otimes_k C^* \cong \mathrm{End}_k(C)$ (c.f. the module-theoretic version)

Def: An endotrivial kG-module M satisfies

$$\mathrm{End}_k(M) \cong k \oplus P$$

where P is a projective kG-module.

This is not a generalization, but rather an analogous formulation for the homotopy category.

Thm (M) Let $eTriv(kG)$ denote the set of endotrivial complexes of kG-modules. Then, the following diagram is well-defined and commutes.

$$\begin{tikzcd}
B(G)^\times \arrow[r, hook, "\mathrm{Ind}_{\Delta G}^{G\times G}"] \arrow[d, "k[-]"] & O(B(G,G)) \arrow[d, "k[-]"] \\
O(T(kG)) \arrow[r, hook, "\mathrm{Ind}_{\Delta G}^{G\times G}"] & O(T^\Delta(kG,kG)) \\
eTriv(kG) \arrow[u, "\Lambda(-)"] \arrow[r, hook, "\mathrm{Ind}_{\Delta G}^{G\times G}"] & Spl(kG,kG) \arrow[u, "\Lambda(-)"]
\end{tikzcd}$$

<u>Pf</u> (idea) Need to show:

(1) For C, D chain complexes of kG-modules,

$\mathrm{Ind}_{\Delta G}^{G\times G}(C \otimes_k D) \cong \mathrm{Ind}_{\Delta G}^{G\times G}(C) \otimes_{kG} \mathrm{Ind}_{\Delta G}^{G\times G}(D)$ naturally

(2) For $H \leq G$, C a complex of kH-modules,

$\mathrm{Ind}_H^G(C^*) \cong (\mathrm{Ind}_H^G(C))^*$ naturally

(3) $\mathrm{Ind}_{\Delta G}^{G\times G}(C)^*$ is "well-defined"

- $(-)^*$ here could refer to either the left R-module dual or the bimodule dual, and in general these do not coincide.

  However in this case they are naturally isomorphic! ▨

<u>TL;DR</u>: it suffices to construct endotrivial complexes to find splendid Rickard complexes, and they associate to units of $B(G)$ in the same way.

Pros of endotrivial modules:

- Easier to do computations with (smaller k-dimension)
- Inflation of units of $B(G)$ ⟷ Inflation of endotrivial complexes!
- Can use the Künneth formula to compute homology

Con: less generality - not all p-permutation equivalences arise from $O(T(kG))$

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(3) $B(G)^\times$ for p-groups

Through the use of Biset functors, Bouc gave a complete description of $B(G)^\times$ for any p-group G.

We will use this by creating an analogous construction for endotrivial complexes.

If p is odd, $B(G)^\times$ is easy to describe.

<u>Thm:</u> (tom Dieck)

The Feit-Thompson theorem (groups of odd order are solvable) is equivalent to the statement that "if G has odd order then $B(G)^\times = \{\pm[G/G]\}$"

So while $B(G)^\times$ is easy to describe, it is highly non-trivial to prove the description! (Feit-Thompson's proof is around 100 pages)

In the case of $p=2$, the classification of $B(G)^\times$ for $|G|=2^k$ is described inductively. We'll give an approximation of the theorem

(Recall $B(G)^\times$ is always an elementary abelian 2-group, so it suffices to describe its generators.)

<u>Thm</u> (Bouc) Let G be a 2-group.

The generators of $B(G)^\times$ which are <u>not</u> of the form

$$\mathrm{Inf}^G_{G/N} u \qquad (1 < N \trianglelefteq G,\ u \in B(G/N)^\times)$$

have the following form: consider the set of of subgroups

$$\mathcal{H} = \{ S < G : N_G(S)/S \cong C_2 \text{ or } D_{2^n},\ n \geq 4 \}$$

For each $S \in \mathcal{H}$, the following is a generator of $B(G)^\times$:

- If $N_G(S)/S \cong C_2$

$$u = \mathrm{Ten}^G_{N_G(S)} \mathrm{Inf}^{N_G(S)}_{N_G(S)/S} \left([C_2/C_2] - [C_2/1] \right)$$

- If $N_G(S)/S \cong D_{2^n}$

$$u = \mathrm{Ten}^G_{N_G(S)} \mathrm{Inf}^{N_G(S)}_{N_G(S)/S} \left([D_{2^n}/D_{2^n}] - [D_{2^n}/H_1] - [D_{2^n}/H_2] + [D_{2^n}/1] \right)$$

where H_1, H_2 are non-conjugate non-central subgroups of order 2.

All such <u>faithful</u> generators arise in this way.

Here, Ten refers to tensor induction. To be discussed soon...

We have units, so lets lift them!

<u>Thm</u> (M.) The fundamental units above for $B(C_2)^\times$, $B(D_{2^n})^\times$ have associate endotrivial complexes of $\mathbb{Z}G$-modules

- C_2: $\quad 0 \to \mathbb{Z}C_2 \to \mathbb{Z} \to 0$

- D_{2^n}: $\quad 0 \to \mathbb{Z}D_{2^n} \to \mathbb{Z}[D_{2^n}/H_1] \oplus \mathbb{Z}[D_{2^n}/H_2] \to \mathbb{Z} \to 0$

Why tensor induction instead of the usual induction?

- Usual induction is additive but not multiplicative:

$$\mathrm{Ind}^G_H(M) \oplus \mathrm{Ind}^G_H(N) \cong \mathrm{Ind}^G_H(M \oplus N)$$

$$\mathrm{Ind}^G_H(M) \otimes_R \mathrm{Ind}^G_H(N) \ncong \mathrm{Ind}^G_H(M \otimes_R N)$$

(the same applies in $B(G)$ for $\sqcup, \times$)

- Tensor induction is multiplicative but not additive:

$$\mathrm{Ten}^G_H(M) \oplus \mathrm{Ten}^G_H(N) \ncong \mathrm{Ten}^G_H(M \oplus N)$$

$$\mathrm{Ten}^G_H(M) \otimes_R \mathrm{Ten}^G_H(N) \cong \mathrm{Ten}^G_H(M \otimes_R N)$$

So usual induction $\mathrm{Ind}^G_H : B(H) \to B(G)$ will not restrict to units, but tensor induction $\mathrm{Ten}^G_H : B(H)^\times \to B(G)^\times$ will.

(4) Tensor induction of modules and complexes

What exactly is tensor induction anyways?

We'll define it first for modules, then try to build an analogous version for chain complexes.

Idea: since all units arise from teninf-ing the "fundamental" units, an associate endotrivial complex should arise from teninf-ing the corresponding fundamental complex.

Let H be a group. S_n acts on $H^n = H \times \cdots \times H$ by:

$$\pi \cdot (h_1, \dots, h_n) = (h_{\pi^{-1}(1)}, \dots, h_{\pi^{-1}(n)})$$ (h_i is sent to the $\pi(i)$th component)

So we may form the semidirect product $H^n \rtimes S_n$.

This is the wreath product of n copies of H, denoted $H \wr n$.

Given a kH-module M, we can form the $k[H \wr n]$-module $M \wr n$,

$$M \wr n = M \otimes_k \cdots \otimes_k M \quad \text{(as } kH^n\text{-module)}$$

$$(h_1, \dots, h_n; \pi) \cdot m_1 \otimes \cdots \otimes m_n = h_1 m_{\pi^{-1}(1)} \otimes \cdots \otimes h_n m_{\pi^{-1}(n)}$$

Now, if $H \leq G$, $[G:H] = n$, then there is a (noncanonical) embedding

$$G \hookrightarrow H \wr n$$

as follows. Choose coset reps $[G/H] = \{g_1, \dots, g_n\}$, then there is a $\pi \in S_n$ and $h_i \in H$ such that

$$g g_i = g_{\pi(i)} h_i.$$

The embedding is defined by

$$g \longmapsto \pi \cdot (h_1, \dots, h_n; 1) = (h_{\pi^{-1}(1)}, \dots, h_{\pi^{-1}(n)}; \pi)$$

This embedding defines tensor induction:

$$\mathrm{Ten}_H^G M = \mathrm{Res}_G^{H \wr n} M \wr n$$

(it follows that the isomorphism class is independent of choice of coset representatives, and all embeddings are H^n-conjugate)

Lets try applying this idea to chain complexes!

For a kH-chain complex $C = (0 \to C_n \xrightarrow{d_n} \cdots \xrightarrow{d_1} C_0 \to 0 \to \cdots)$ set

$$C \wr n = C \otimes_k \cdots \otimes_k C$$

as a chain complex of kH^n-modules.

Then, define the $k[H \wr n]$-module structure via, for

$$m_1 \otimes \cdots \otimes m_n \in C_{a_1} \otimes_k \cdots \otimes_k C_{a_n} \subseteq (C \wr n)_{a_1 + \cdots + a_n}$$

$$(h_1, \dots, h_n; \pi) m_1 \otimes \cdots \otimes m_n = h_1 m_{\pi^{-1}(1)} \otimes \cdots \otimes h_n m_{\pi^{-1}(n)} \in C_{a_{\pi^{-1}(1)}} \otimes_k \cdots \otimes_k C_{a_{\pi^{-1}(n)}} \subseteq (C \wr n)_{a_1 + \cdots + a_n}$$

Seems fine, except for one thing...

The transition maps of $C \otimes_k \cdots \otimes_k C$ are kH^n-linear, but NOT $k[H \wr n]$-linear, by the differential graded structure.

In some instances, the diagrams which need to commute are off by a sign. Can we fix this by adding a sign to the S_n-action?

Yes - Evens, in construction of the Evens norm map on Ext, determined the correct signage.

For $m_1 \otimes \cdots \otimes m_n \in C_{a_1} \otimes_k \cdots \otimes_k C_{a_n} \subseteq (C \wr n)_{a_1 + \cdots + a_n}$,

$$\pi \cdot m_1 \otimes \cdots \otimes m_n = (-1)^{\nu} m_{\pi^{-1}(1)} \otimes \cdots \otimes m_{\pi^{-1}(n)}$$

$$\nu = \sum_{\substack{j<k \\ \pi(j) > \pi(k)}} a_j a_k$$

i.e., write π as a product of standard transpositions $(1, i)$ and for each one, multiply by $(-1)^{a_j a_k}$.

Then, we define $\mathrm{Ten}_H^G C_\cdot = \mathrm{Res}_G^{H \wr n} C \wr n$ as before.

This construction satisfies the usual properties of tensor induction, which we need for our grand finale.

Prop (M.) Let $K \leq H \leq G$ be groups with finite index, $[H:K]=m$, $[G:H]=n$, C, D bounded chain complexes of kK-modules.

(1) $\mathrm{Res}^{K \wr mn}_{(K \wr m)\wr n} C \wr mn \cong (C \wr m) \wr n$

(2) $\mathrm{Ten}^G_H \mathrm{Ten}^H_K C \cong \mathrm{Ten}^G_K C$

(3) $\mathrm{Ten}^H_K (C^*) \cong (\mathrm{Ten}^H_K C)^*$ and $(C \wr m)^* \cong (C^*) \wr m$

(4) $(C \wr m) \otimes_k (D \wr m) \cong (C \otimes_k D) \wr m$ and $\mathrm{Ten}^H_K C \otimes_k \mathrm{Ten}^H_K D \cong \mathrm{Ten}^H_K (C \otimes_k D)$

However... this is where the fairytale ends!

(5) Things that break

(1) Tensor induction does not preserve homotopy equivalence or contractibility. (but sometimes it does!)

Ex: $\mathrm{ten}^{C_2}_1 (0 \to \mathbb{Z} \to \mathbb{Z} \to 0) = 0 \to \mathbb{Z}^- \to \mathbb{Z}C_2 \to \mathbb{Z} \to 0$

Recall: Endotrivial complexes must satisfy $C \otimes_k C^* \simeq k$

But in general,

$\mathrm{ten}^G_H (C) \otimes_k \mathrm{ten}^G_H (C)^* \cong \mathrm{ten}^G_H (C \otimes_k C^*) \not\simeq \mathrm{ten}^G_H k \cong k$

(2) The components of a tensor induced complex do not in general correspond to those of a tensor induced (virtual) module.

More precisely, the following diagram does not always commute

$$\begin{tikzcd} e\mathrm{Triv}(kH) \arrow[r, "\mathrm{ten}_H^G"] \arrow[d, "\Lambda(-)"'] & e\mathrm{Triv}(kG) \arrow[d, "\Lambda(-)"] \\ T(kH) \arrow[r, "\mathrm{ten}_H^G"'] & T(kG) \end{tikzcd}$$

– tensor induction of chain complexes

– tensor induction of virtual modules

<u>2 reasons why:</u>

(1) The sign change in the chain complex construction

(2) The virtual module construction is computed via Dress's theory of algebraic maps, rather than simply extending to positive and negative components of a virtual module.

However if $\mathrm{char}(k) = 2$ and the complex satisfies some nice properties, the diagram does commute. In particular, the endotrivial complexes given for the fundamental units allow the diagram to commute!

(3) Over $\mathcal{O}$, ten_H^G does not send p-permutation chain complexes to p-permutation chain complexes.

(For simplicity I have only worked over k, but splendid equivalences were originally defined over $\mathcal{O}$. However, Rickard proved that SRCs over $\mathcal{O}$ and k correspond bijectively.)

Instead, ten_H^G sends linear source chain complexes to linear source chain complexes over $\mathcal{O}$.

On the other hand, over k, ten_H^G indeed is closed under p-permutation chain complexes.

Next steps and Closing thoughts

- Tensor induction on complexes appears unstudied – this construction has applications to transfer, so understanding properties it has could be useful.

- If I have a criteria for when $\mathrm{ten}_H^G(0 \to k[H/K] \to k[H/K] \to 0)$ is contractible for $\mathrm{char}(k) = 2$, I will be able to say when tensor induction sends endotrivial complexes to themselves for 2-groups. I currently have a sufficient condition if $K \trianglelefteq H$.

- All work I've done has related to permutation modules. Boltje and Carman recently characterized $T(kG)$ which should provide some clarity for the non-permutation case, but new methods will likely be needed.

- Separate, potentially interesting and useful question: how to detect if a homomorphism of (potentially large) permutation modules splits (injectively/surjectively to start)? Motivated by my proof that

$$0 \longrightarrow \mathbb{Z}D_{2^n} \longrightarrow \mathbb{Z}[D_{2^n}/H_1] \oplus \mathbb{Z}[D_{2^n}/H_2] \longrightarrow \mathbb{Z} \longrightarrow 0$$

is endotrivial.

(which is by far the thing that took the most time to show in my research so far)