Endotrivial complexes
(and the trivial source ring)

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Conventions & Notation:

- G is a finite group.
- p is a prime.
- k is a field of characteristic p.
- $s_p(G)$ denotes the set of all p-subgroups of G.
- All kG-modules are finitely generated.
- ${}_{kG}\textbf{mod}$ denotes the category of all finitely generated kG-modules.

Definition (p-permutation module)

- A kG-module M is a *p-permutation module* if for all $P \in s_p(G)$, $\mathrm{res}^G_P M$ is a permutation module. Equivalently, M is a direct summand of a permutation module.
- ${}_{kG}\mathbf{triv}$ is the category of p-permutation modules.

Example

- If $p \nmid |G|$, ${}_{kG}\mathbf{triv} = {}_{kG}\mathbf{mod}$.
- If G is a p-group, p-permutation modules are permutation modules.

Theorem (Balmer, Gallauer '22)

$K^b({}_{kG}\mathbf{triv})/K^b_{ac}({}_{kG}\mathbf{triv}) \cong D^b({}_{kG}\mathbf{mod})$.

Definition (Brauer construction)

- Given $M \in {}_{kG}\textbf{mod}$ and $P, Q \in s_p(G)$ with $Q < P \leq G$, the *trace map* $\mathrm{tr}_Q^P : M^Q \to M^P$ is
$$\mathrm{tr}_Q^P(m) = \sum_{p \in [P/Q]} p \cdot m.$$
- Given any p-subgroup $P \leq G$, the *Brauer construction* is the additive functor $-(P) : {}_{kG}\textbf{mod} \to {}_{k[N_G(P)/P]}\textbf{mod}$ induced by:
$$M(P) := M^P / \sum_{Q<P} \mathrm{tr}_Q^P M^Q.$$

Example

If X is a G-set, then $(k[X])(P) \cong k[X^P]$ as $k[N_G(P)/P]$-modules.

The Brauer construction is very well-behaved with p-permutation modules. It is responsible for many "local-to-global" phenomena.

Properties

Let $M, N \in {}_{kG}\textbf{triv}$.

- $M(P) \in {}_{k[N_G(P)/P]}\textbf{triv}$.
- $M(P) \otimes_k N(P) \cong (M \otimes_k N)(P)$ for all $P \in s_p(G)$.
- $M(P)^* \cong M^*(P)$ for all $P \in s_p(G)$.
- $f : M \to N$ is split injective (resp. surjective) if and only if $f(P) : M(P) \to N(P)$ is injective (resp. surjective) for all $P \in s_P(G)$.

...and many more, but the complete list does not fit within the margins of this talk.

Definition (Endotrivial complexes)

- A bounded chain complex $C \in Ch^b({}_{kG}\mathbf{triv})$ is *endotrivial* if

$$\mathrm{End}_k(C) \cong C^* \otimes_k C \simeq k[0],$$

  i.e. $C^* \otimes_k C \cong k \oplus D$ for some contractible chain complex D.
- Let $\mathcal{E}_k(G)$ denote the set of homotopy classes of endotrivial complexes. $(\mathcal{E}_k(G), \otimes_k)$ forms an abelian group.

Side note: another plausible generalization of endotriviality is complexes C for which $C^* \otimes_k C \simeq k$ in $K^b(\underline{{}_{kG}\mathbf{triv}})$.

Theorem (M.)

Let $C \in Ch^b({}_{kG}\textbf{triv})$. The following are equivalent:

- *C is endotrivial.*
- *For every $P \in s_P(G)$, $C(P)$ has nonzero homology in exactly one degree, with that homology having k-dimension 1.*

Proof: (sketch)
$(\implies)$: Let C be endotrivial and P a p-subgroup of G. Then properties of the Brauer construction imply

$$\big(C(P)\big)^* \otimes_k C(P) \cong (C^* \otimes_k C)(P) \simeq k(P) \cong k.$$

The Künneth formula implies the rest.

($\Longleftarrow$) : Suppose for all p-subgroups P, $C(P)$ has nonzero homology in exactly one degree with k-dimension 1. Then the Künneth formula and properties of the Brauer quotient imply that

$$H_0\big((C^* \otimes_k C)(P)\big) \cong k,$$

and $(C^* \otimes_k C)(P)$ is acyclic at all other degrees. Denote the differentials of $C^* \otimes_k C$ by $\{d_i\}$.

- If C has length n, then by homology, $d_{n-1}(P)$ is injective for all p-subgroups P, hence split injective.
- Similarly, $d_{-n+2}(P)$ is surjective for all p-subgroups P, hence split surjective.

Thus $C^* \otimes_k C \simeq D$, for some complex D with two fewer nonzero terms. An inductive argument shaves down D until one nonzero degree remains, and we are done. □

Observe: for any p-subgroup $P \leq G$, the Brauer construction induces a group homomorphism $-(P) : \mathcal{E}_k(G) \to \mathcal{E}_k(N_G(P)/P)$.

Definition (h-marks)

If C is an endotrivial complex, let $h(C)$ denote the degree in which C has nontrivial homology. If $P \in s_p(G)$, say $h(C(P))$ is the *h-mark of C at P*.

Question: How much do "local" homological properties, like the h-marks, determine the structure of an endotrivial complex?

Answer: Pretty much completely.

Theorem (M.)

$$\epsilon : \mathcal{E}_k(G) \to \left(\prod_{P \in s_p(G)} \mathbb{Z} \right)^G$$
$$[C] \mapsto (h(C(P)))_{P \in s_p(G)}$$

is a group homomorphism, with kernel the torsion subgroup of $\mathcal{E}_k(G)$*,*

$$\{k_\omega[0] \mid \omega \in \mathrm{Hom}(G, k^\times)\}.$$

In particular, $\mathcal{E}_k(G)$ *is finitely generated with* $\mathbb{Z}$*-rank bounded by the number of conjugacy classes of* p*-subgroups.*

Proof idea: similar to the previous proof, shave off contractible terms.

Why require p-permutation modules? (besides their friendliness)

Theorem (M.)

Let C be an endotrivial complex of kG-modules. Set

$$\Delta G = \{(g,g) \in G \times G \mid g \in G\}$$

and identify G with ΔG. $\mathrm{ind}_{\Delta G}^{G\times G} C$, regarded as a chain complex of (kG, kG)-bimodules, is a splendid Rickard autoequivalence of kG.

Corollary (M.)

Let $(K, \mathcal{O}, k)$ be a p-modular system large enough for G, and let C be an endotrivial complex of kG-modules. There exists a unique (up to iso.) endotrivial complex $\hat{C}$ of $\mathcal{O}G$-modules such that $k \otimes_{\mathcal{O}} \hat{C} \cong C$.

Question: How can we go about determining generators of $\mathcal{E}_k(G)$?
One approach we take uses ideas from Bouc's theory of biset functors.

Definition (Faithful constituents)

Let $s_p^{\trianglelefteq}(G)$ denote the set of normal p-subgroups of G. The *faithful constituent* of $\mathcal{E}_k(G)$, denoted $\partial\mathcal{E}_k(G)$, is

$$\partial\mathcal{E}_k(G) = \bigcap_{1<P\in s_p^{\trianglelefteq}(G)} \ker(-)(P).$$

$\partial\mathcal{E}_k(G) \leq \mathcal{E}_k(G)$.

Elements of $\partial\mathcal{E}_k(G)$ have nonzero h-marks only at p-subgroups of G which do not contain a nontrivial normal subgroup of G.

Theorem

$$\prod_{P \in s_p^{\trianglelefteq}(G)} \partial\mathcal{E}_k(G/P) \to \mathcal{E}_k(G)$$

$$([C_P])_P \mapsto \bigotimes_{P \in s_p^{\trianglelefteq}(G)} \inf_{G/P}^{G}[C_P]$$

is an isomorphism.

Strategy: Determine $\partial\mathcal{E}_k(G)$ to inductively build up $\mathcal{E}_k(G)$.

Determining $\mathcal{E}_k(G)$ for abelian groups:

- Since every subgroup is normal, $\mathrm{rk}_{\mathbb{Z}}\, \partial\mathcal{E}_k(G) \in \{0,1\}$, corresponding to the h-mark at 1.
- If G has p-rank 0,

$$\partial\mathcal{E}_k(G) = \mathcal{E}_k(G) \cong \{k_\omega[i] \mid \omega \in \mathrm{Hom}(G, k^\times), i \in \mathbb{Z}\}.$$

- If G has p-rank 1, there exists a periodic resolution of the trivial kG-module k by projectives $C \to k$. Truncating such a resolution appropriately gives a faithful endotrivial complex and generates $\partial\mathcal{E}_k(G)$.
- We can show $\partial\mathcal{E}_k(G) = 0$ when G has p-rank > 1 using that:
  - the trivial kG-module has periodic cohomology if and only if G has p-rank 1.
  - a theorem of Bouc which gives a local condition for when a complex of p-permutation modules is homotopy equivalent to a complex of projectives.

Theorem (M.)

Let G be an abelian group. Let $s_p^{(0,1)}(G)$ denote the set of all p-subgroups of G for which G/P has p-rank 0 or 1. Then,

$$\mathcal{E}_k(G) \cong \prod_{P \in s_p^{(0,1)}(G)} \partial\mathcal{E}_k(G/P),$$

with a generating set described by the previous slide.

Example

Other classes of groups for which we have determined generators for $\mathcal{E}_k(G)$:

- Dihedral groups
- Generalized quaternion groups
- p-nilpotent groups with Sylow p-subgroup one of the above

Downside: explicitly constructing faithful endotrivial complexes is difficult. It's easier to prove they can't exist.

Definition (Trivial source ring)

- The *trivial source ring* $T(kG)$ is the Grothendieck group of ${}_{kG}\mathbf{triv}$ with respect to split exact sequences. It forms a ring via $\otimes_k$.
- The *orthogonal unit group* $O(T(kG)) \leq T(kG)^\times$ is the subgroup consisting of units $u \in T(kG)^\times$ for which $u^{-1} = u^*$.

Let $C \in Ch^b({}_{kG}\mathbf{triv})$. The *Lefschetz invariant* of C is

$$\Lambda(C) = \sum_{i\in\mathbb{Z}} (-1)^i [C_i] \in T(kG).$$

Proposition

Let $C \in Ch^b({}_{kG}\mathbf{triv})$ be an endotrivial complex. Then $\Lambda(C) \in O(T(kG))$.

Question: What is the cokernel of $\Lambda(-) : \mathcal{E}_k(G) \to O(T(kG))$?

Strategy: Consider local homology.

(Boltje & Carman '23) showed that every $u \in O(T(kG))$ can be uniquely expressed as a signed character tuple $(\epsilon_P \cdot \rho_P)_{P \in s_p(G)}$, with

- $\epsilon_P \in \{\pm 1\}$
- $\rho_P \in \mathrm{Hom}(N_G(P)/P, k^\times)$, i.e. a degree 1 character.

$\epsilon_P \cdot \rho_P$ is the image of $u(P) \in O(T(k[N_G(P)/P]))$ in the Brauer character ring $R_k(N_G(P)/P)$, i.e. the Grothendieck group of ${}_{kG}$**mod** with respect to *short* exact sequences.

The Frobenius automorphism $F : k \to k$, $x \mapsto x^p$, induces a ring automorphism on kG, hence an exact endofunctor on ${}_{kG}\textbf{mod}$ and ${}_{kG}\textbf{triv}$. This induces group automorphisms on $O(T(kG))$ and $\mathcal{E}_k(G)$.

Given a kG-module M, let ${}^F M$ be the kG-module which results by twisting by F.

Example

If $\omega \in \mathrm{Hom}(G, k^\times)$, then ${}^F k_{F\circ\omega} \cong k_\omega$ as kG-modules.

It turns out endotrivial complexes have a F-stability condition!

Theorem (M.)

Let $u \in O(T(kG))$ have a corresponding character tuple (ρ_P), and suppose ρ_1 is the trivial character. If u has an endotrivial lift $C \in \mathcal{E}_k(G)$, then $\rho_P = {}^F\rho_P$ for all $P \in s_p(G)$. Equivalently, all ρ_P are $\mathbb{F}_p^\times$-valued.

Proof idea: Observe that twisting an endotrivial complex by F preserves h-marks and acts pointwise on local homology. Then cook up a contradiction using the characterization of $\ker \epsilon$. □

This easily extends to the other orthogonal units by twisting by appropriate 1-dimensionals.

Consequences:

- If $p = 2$, the only orthogonal units which can lift arise from units of Burnside rings and $\pm$ 1-dimensionals.
- If $p > 2$ and all primes q which divide $|G|$ satisfy $q \nmid p - 1$, the only units which lift are $\pm$ 1-dimensionals.

Final remarks and ongoing work:

- Determining the cokernel of $\Lambda(-)$ for 2-groups in general seems very hard.
- Tensor induction seems like it should preserve endotriviality, but it generally doesn't :(
- If $S \in \mathrm{Syl}_p(G)$, $\mathrm{res}^G_S : \mathcal{E}_k(G) \to \mathcal{E}_k(S)$ does not lose any h-mark information! Can we characterize the cokernel? Yes for p-nilpotent groups. (this is analogous to work of Barsotti & Carman for Burnside rings)
- How many of these ideas can be generalized to a a "stable w.r.t. $\mathcal{X}$-projectives" endotrivial setting?
- **Thank you for your time!!** - sam :)