A Splendid Lift of Equivalences

Lifting p-Permutation Equivalences to Splendid Rickard Complexes

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Outline

1. Representation-theoretic preliminaries.
   Goal: introduce **vertices, sources, p-permutation modules,** and **blocks**.
2. Category theory and equivalences of categories.
   Goal: introduce **Splendid Rickard Complexes** and **Broue's Abelian Defect Group Conjecture**.
3. Grothendieck groups of p-permutation modules.
   Goal: introduce p-**permutation equivalences** and their connection to splendid Rickard complexes.
4. Burnside rings.
   Goal: explain how units of $B(G)$ induce p-permutation equivalences.
5. Discuss my initial work.
   Goal: sketch the proof that all p-permutation equivalences arising from units of the form $[G/G] - [G/H]$ lift to splendid Rickard complexes.

For this presentation, a representation of G over a field k is a finitely generated left kG-module, i.e. an object in ${}_{kG}\textbf{mod}$.

Definition

A kG-module M is:

- **irreducible** or **simple** if its only submodules are $\{0\}$ and M.
- **indecomposable** if M cannot be written as $M = M_1 \oplus M_2$ for two nonzero representations M_1, M_2.
- **projective** if M is a direct summand of a free module (equivalently a projective object in ${}_{kG}\textbf{mod}$).
- a **permutation representation** if M has a G-stable k-basis, i.e. $M \cong k[X]$ for a G-set X.

In **modular** representation theory, we consider representations over fields of prime characteristic which divide the order of G.

Representations over $\mathbb{C}$

- Simple $\iff$ indecomposable.
- Finitely many simple/indecomposable representations.
- All representations decompose into a direct sum of simple representations.
- All representations are projective, i.e. $\mathbb{C}G$ is semisimple.

Representations over Prime Characteristic Dividing $|G|$

- Simple $\not\iff$ indecomposable.
- There are, in general, an infinite number of indecomposable representations.
- Representations do not in general decompose into simple representations.
- Almost all indecomposable representations are not projective - in fact, kG has infinite global projective dimension.

Definition (p-Modular Systems)

A p-**modular system large enough for** G is a set of three commutative rings $(K, \mathcal{O}, k)$, where:

1. $\mathcal{O}$ is a complete discrete valuation ring that has a $\exp(G)$th root of unity.
2. K is the the field of fractions of $\mathcal{O}$ and has characteristic 0.
3. k is the residue field of $\mathcal{O}$ and has characteristic p.

We study KG-, $\mathcal{O}G$-, and kG-modules. The theory of representations over K is the same as the classical case of representations over $\mathbb{C}$. It is a theorem that every perfect field F of characteristic p has a corresponding p-modular system $(K, \mathcal{O}, F)$ with p prime in $\mathcal{O}$.

For the rest of this presentation, we will assume p is prime, $(K, \mathcal{O}, k)$ is a p-modular system large enough for G, and unless specified, let $R \in \{\mathcal{O}, k\}$.

Example

Let ζ be a $\exp G$-th root of unity. A p-modular system large enough for G is

- $K = \mathbb{Q}_p(\zeta)$.
- $\mathcal{O} = \mathbb{Z}_p[\zeta]$.
- $|k| = p^f$ for some $f \in \mathbb{N}^+$.

Let $H \leq G$ be groups.

Induction and Restriction

1. Given a RG-module M, it can be viewed as an RH-module by restricting scalars to $RH \subseteq RG$. This is denoted $\operatorname{Res}^G_H M$.
2. Given a RH-module M, it can be extended to a RG-module by extending scalars to RG. Formally,

$$\operatorname{Ind}^G_H M := RG \otimes_{RH} M.$$

Theorem (Krull-Schmidt Theorem for Modules)

Let R be a ring for which one of the two hold:

1. *R is complete, commutative, local, and Noetherian, and let A be a R-algebra that is finitely generated as an R-module,*
2. *R is a field, and let A be a finite dimensional R-algebra,*

and let M be a finitely generated A-module. If $M = \bigoplus_{i=1}^m M_i = \bigoplus_{j=1}^n N_j$ are two decompositions of M into indecomposable summands, then $m = n$ and there is a permutation $\pi \in S_n$ such that $M_i \cong N_{\pi(i)}$ for all $i \in \{1, \ldots, n\}$.

In particular, the Krull-Schmidt theorem for modules holds for $(K, \mathcal{O}, k)$.

Definition (Relative Projectivity)

Say a RG-module M is **relatively H-projective** if there exists a RH module N such that M is isomorphic to a direct summand of $\mathrm{Ind}_H^G N$ (writ. $M \mid \mathrm{Ind}_H^G N$).

Let $K \leq H \leq G$. Because $\mathrm{Ind}_K^G = \mathrm{Ind}_H^G \circ \mathrm{Ind}_K^H$, if M is K-projective, it is also H-projective.

By the Krull-Schmidt theorem for modules, this notion is well-defined.

Definition (Vertices and Sources)

If M is an indecomposable RG-module, then there are minimal p-subgroups $P \leq G$ such that M is P-projective. In fact, all such P form a complete conjugacy class of p-subgroups of G.

1. Any such P is a **vertex** of M.
2. In this case, one may find an indecomposable RP-module S such that $M \mid \mathrm{Ind}_P^G S$. Any such S is a P-**source** of M.

Any pair (P, S) is a **vertex-source pair** of M.

Trivial Source Modules

Indecomposable RG-modules with trivial source (a **trivial source module**) are of special interest.

1. M has trivial source if and only if it is a direct summand of a permutation module.
2. If M is a trivial source $\mathcal{O}G$-module with vertex P, then the kG-module $\overline{M} := k \otimes_{\mathcal{O}} M$ is a trivial source kG-module with vertex P.
3. Conversely, if N is a trivial source kG-module with vertex P, there is up to isomorphism a *unique* trivial source $\mathcal{O}G$-module M with vertex P such that $N \cong \overline{M}$.

p-Permutation Modules

We say a RG-module M is a p-**permutation module** if $\operatorname{Res}^G_P M$ is a permutation module for some Sylow p-subgroup $P \leq G$.

1. M is a p-permutation module if and only if it is a direct sum of trivial source modules.
2. By the facts stated in the previous block, isomorphism classes of p-permutation $\mathcal{O}G$-modules and p-permutation kG-modules correspond bijectively.

A primitive idempotent e is an idempotent which cannot be written $e = e_1 + e_2$ for two orthogonal idempotents e_1, e_2, i.e. $e_1e_2 = e_2e_1 = 0$.

Definition (Blocks)

The unique decomposition of $1 = e_1 + \cdots + e_k$ into primitive idempotents of $Z(\mathcal{O}G)$ gives a decomposition

$$\mathcal{O}G = \mathcal{O}Ge_1 \oplus \cdots \oplus \mathcal{O}Ge_k$$

of $(\mathcal{O}G, \mathcal{O}G)$-bimodules (or as a $\mathcal{O}$-algebra, as a direct product). This is the *unique* decomposition of $\mathcal{O}G$ into a direct sum of two-sided ideals of $\mathcal{O}G$. We call the $\mathcal{O}Ge_i$s the **block (algebras)** of $\mathcal{O}G$, and the e_is are the **block (idempotents)** of $\mathcal{O}G$. The blocks are again $\mathcal{O}$-algebras.

The block idempotents are also (not necessarily primitive) idempotents of $Z(KG)$, giving a decomposition

$$KG = KGe_1 \oplus \cdots \oplus KGe_k,$$

and the reduction $\overline{e_i} \in Z(kG)$ gives the primitive idempotents of $Z(kG)$ and a complete decomposition

$$kG = kG\overline{e_1} \oplus \cdots \oplus kG\overline{e_k}.$$

Belonging to Blocks

Let RG have block idempotents $\{e_1, \cdots, e_k\}$. Then any RG-module M has direct sum decomposition

$$M = Me_1 \oplus \cdots \oplus Me_k.$$

If $Me_i = M$ for some block e_i, then we say M_i "belongs to e_i."
In particular, every indecomposable RG-module belongs to a unique block.

Definition (Principal Block)

The block that the **trivial representation** $M = R$ belongs to is called the **principal block**. Equivalently, it is the block e for which $I(e) = 1$, where $I : RG \to R$ is the augmentation homomorphism of RG.

Bimodules

Let G and H be groups, R any ring, and suppose M is a (RG, RH)-bimodule. Then M can equivalently be considered a $R[G \times H]$-module or $RG \otimes_R RH$-module induced by the formula

$$g \cdot m \cdot h^{-1} \leftrightarrow (g,h) \cdot m \leftrightarrow (g \otimes h) \cdot m.$$

Under this identification, the previous notions discussed (vertices, sources, blocks) apply to bimodules as well. In particular, we have trivial source bimodules and p-permutation bimodules.

Definition (Defect Group)

Each block RGe can also be regarded as a (RG, RG)-bimodule, hence a $R(G \times G)$-module. It has vertex $\Delta P \leq G \times G$ for a p-subgroup $P \leq G$, since $RG \cong \mathrm{Ind}_{\Delta G}^{G \times G}(R)$.

We say P is a **defect group** of the block e, and we say the **defect** of e is the unique integer d such that $|P| = p^d$.

We begin by motivating splendid Rickard equivalences with Morita's theorem. Let A be a R-algebra (for example, RG).

Notation

Let A, B be R-algebras and $\mathcal{C}$ be an abelian category. We set notation for the following categories:

- ${}_A\textbf{Mod}$ is the category of A-modules, and ${}_A\textbf{mod}$ is the full subcategory of finitely generated A-modules.
- ${}_A\textbf{Mod}_B$ is the category of (A, B)-bimodules, and ${}_A\textbf{mod}_B$ is the full subcategory of finitely generated (A, B)-bimodules.
- $K(\mathcal{C})$ is the homotopy category of $\mathcal{C}$, and $K^b(\mathcal{C})$ is the full subcategory consisting of bounded chain complexes.
- $D(\mathcal{C})$ is the derived category of $\mathcal{C}$, and $D^b(\mathcal{C})$ is the full subcategory consisting of bounded chain complexes.

If $A \cong B$, then there is an equivalence of categories ${}_A\textbf{mod} \cong {}_B\textbf{mod}$ as abelian categories. However, the converse is not always true.

Example

${}_A\textbf{mod} \cong {}_{M_n(A)}\textbf{mod}$ for any positive integer n. A functor is induced by tensoring by the bimodules $M_n(A)e$ and $eM_n(A)$, where e is the primitive idempotent of $M_n(e)$ with a 1 in the $(1, 1)$th coordinate.

Kiiti Morita proved that any such equivalence of categories arises via tensoring by suitable bimodules.

Theorem (Morita)

*${}_A$**Mod** $\cong$ ${}_B$**Mod** if and only if there exist an (A,B)-bimodule M and a (B,A)-bimodule N such that*

- *$M \otimes_B N \cong A$ as (A,A)-bimodule,*
- *$N \otimes_A M \cong B$ as (B,B)-bimodules,*

and with M and N finitely generated projective as left and right modules.

*In this case, the functors $M \otimes_B -$ and $N \otimes_A -$ induce the equivalences of categories, and further, the functors restrict to an equivalence of categories ${}_A$**mod** $\cong$ ${}_B$**mod**.*

An equivalence of module categories of this form is a **Morita equivalence**.

Definition (Symmetric Algebras)

An R-algebra A is **symmetric** if A is finitely generated projective as a R-module and $\mathrm{Hom}_R(A,R) \cong A$ as an (A,A)-bimodule.
Group algebras and block algebras are both symmetric (in fact, if A is symmetric, eAe is symmetric for any idempotent e of A).

Jeremy Rickard introduced an analogue of Morita equivalences for $K^b({}_A\textbf{Mod})$.

Definition (Split Endomorphism Two-Sided Tilting Complex)

If there exists a chain complex of finitely generated (A, B)-bimodules M and a chain complex of finitely generated (B, A)-bimodules N such that:

- $M \otimes_B N \simeq A$,
- $N \otimes_A M \simeq B$,

then $M \otimes_B -$ and $N \otimes_A -$ induce an equivalence of categories $K^b({}_A\textbf{Mod}) \cong K^b({}_B\textbf{Mod})$.

Given an (A, B)-bimodule M, its dual $M^* := \text{Hom}_R(M, R)$ is a (B, A)-bimodule under the actions

$$(b \cdot f \cdot a)(m) = f(a \cdot m \cdot b).$$

This induces a functor

$$(-)^* : {}_A\textbf{Mod}_B \to {}_B\textbf{Mod}_A.$$

If A and B are symmetric, it is possible to find a chain complex M such that the pair (M, M^*) satisfies the above conditions. In this case, we say M is a **(Split Endomorphism) Two-Sided Tilting Complex**.

Rickard proved that equivalences on bounded derived categories can be realized by split endomorphism two-sided tilting complexes, and thus are the same as equivalences on bounded homotopy categories.

Theorem (Rickard)

Let A and B be symmetric R-algebras. The following are equivalent:

1. $D^b({}_A\mathbf{Mod}) \cong D^b({}_B\mathbf{Mod})$.
2. *There exists a two-sided tilting complex of (A, B)-bimodules, and moreover, the complex can be chosen such that all but one of the components are projective as (A, B)-bimodules.*

Further, he and Michel Broué posed that it is reasonable to impose additional conditions:

Definition (Splendid Rickard Complex)

Let A and B be direct summands of RG and RH respectively and X a two-sided tilting complex. Say X is a **Splendid Rickard Complex** if additionally, all the terms of X are p-permutation bimodules with "twisted diagonal" vertices, i.e. of the form ΔP for a common subgroup $P \leq G, H$.
A **Splendid Rickard Equivalence between** A **and** B is the equivalence $K^b({}_A\mathbf{Mod}) \cong K^b({}_B\mathbf{Mod})$ or $D^b({}_A\mathbf{Mod}) \cong D^b({}_B\mathbf{Mod})$ induced by tensoring by X.

Splendid stands for: "**SPL**it-**END**omorphism two-sided tilting complex of summands of permutation modules **I**nduced from **D**iagonal subgroups."

Why p-Permutation Modules?

1. The Brauer construction (a functor ${}_{\mathcal{O}G}\textbf{mod} \rightarrow_{kN_G(P)} \textbf{mod}$) applied to a splendid Rickard complex induces splendid Rickard equivalences between block algebras of "local subgroups," centralizers of subgroups of G and H.
2. The above collection of splendid Rickard equivalences induces an **isotypy**, a compatible collection of "perfect isometries" between character rings of local subgroups.
3. In the case where G is a finite reductive group, the equivalence should be related to the l-adic cohomology of Deligne-Lusztig varieties.

A homotopy equivalence or derived equivalence implies the following for block algebras:

- Isomorphic centers.
- Isomorphic Hochschild cohomology rings.
- Isomorphic "cde" triangles.

If X is a splendid Rickard complex for two block algebras $\mathcal{O}A$ and $\mathcal{O}B$, it is straightforward to show that $k \otimes_{\mathcal{O}} -$ induces a splendid Rickard complex $k \otimes_{\mathcal{O}} X$ for the block algebras kA and kB.

Since p-permutation kG-modules lift uniquely to $\mathcal{O}G$-modules, one may ask if a similar statement holds for splendid Rickard complexes. Indeed, a theorem of Rickard demonstrates that splendid Rickard equivalences between kA, kB, and $\mathcal{O}A$, $\mathcal{O}B$ are in bijection.

Theorem (Rickard)

Given a splendid Rickard complex Y for kA, kB over k, there is a unique splendid Rickard complex X for $\mathcal{O}A, \mathcal{O}B$ such that $Y = k \otimes_{\mathcal{O}} X$.

Practically, this means that when searching for splendid Rickard complexes, it is permissible to look for complexes over k or $\mathcal{O}$.

One motivation to study derived equivalences comes from conjectures of Michel Broué.

Broué's Abelian Defect Group Conjecture

Let G be a finite group and $(K, \mathcal{O}, k)$ a p-modular system large enough for G. Let A be a block algebra of $\mathcal{O}G$, D the defect group of A, and set $H := N_G(D)$. Then by Brauer's first main theorem, there is a corresponding block algebra B of $\mathcal{O}H$ with the same defect group, obtained by applying the Brauer construction.

Conjecture: If D is abelian, then there exists a splendid Rickard equivalence between A and B.

A weaker formulation is as follows:

Alternative Formulation

Suppose G has an abelian Sylow p-subgroup. Set $H := N_G(P)$. Then there is a splendid Rickard equivalence between the principal blocks of $\mathcal{O}G$ and $\mathcal{O}H$.

The best evidence of the existence of these equivalences is the existence of perfect isometries and isotypies between the corresponding blocks - these have been shown to exist in many cases. A splendid Rickard complex would be the "missing link" which provides the corresponding derived category equivalences.

Motivated by Broué's conjecture, Robert Boltje and Philipp Perepelitsky introduced the notion of a p-permutation equivalence, an equivalence on the Grothendieck group level.

Definition (Grothendieck Group of p-permutation modules)

The **Grothendieck Group** $T(RG)$ **of** G (with respect to direct sums) is the quotient of the free abelian group $\mathbb{Z}[I]$, with I the collection of isomorphism classes of p-permutation RG-modules, by the subgroup S which is $\mathbb{Z}$-linearly generated by elements of the form

$$[M] + [N] - [M \oplus N].$$

Restriction to Blocks

1. The standard basis of $T(RG)$ is given by all indecomposable trivial source RG-modules.
2. If e is a block idempotent of RG, we can analogously define $T(RGe)$ and view it as a subgroup of $T(RG)$, via the projection $\omega \mapsto \omega e$. It has standard basis given by all indecomposable trivial source RG-modules belonging to e.
3. Further, we can see $T(RGe_1 \oplus \cdots \oplus RGe_k) = T(RGe_1) \oplus \cdots \oplus T(RGe_k)$. If $A = RGe_1 \oplus \cdots \oplus RGe_k$ is a direct summand of RG, we write $T(A)$ as shorthand for above.

Definition (Grothendieck Group of p-Permutation Bimodules)

To define the Grothendieck group of p-permutation bimodules, we use the aforementioned identification

$$T(RG, RH) := T(R[G \times H]).$$

If e and f are block idempotents of RG and RH respectively, set

$$T(RGe, RHf) := T(R[G \times H](e \otimes \overline{f})),$$

where $\overline{(-)}$ is the involution homomorphism on a group algebra induced by $g \mapsto g^{-1}$. This follows since e is a block idempotent of RG and f is a block idempotent of RH if and only if $e \otimes \overline{f}$ is a block idempotent of $RG \otimes_R RH$.

If A is a direct summand of RG and B a direct summand of RH, we define $T(A, B)$ by taking the direct sum of Grothendieck groups over all pairs of blocks in A and B.

Bilinear Maps

Let A, B, C be direct summands of group algebras. Tensor products of trivial source modules are again trivial source modules, so the tensor product induces a bilinear map

$$T(A,B) \times T(B,C) \to T(A,C), \quad (m,n) \mapsto m \cdot_B n,$$

where $\cdot_B$ is the map induced by $\otimes_B$. Letting $C = R$ shows that any $m \in T(A,B)$ induces a group homomorphism

$$I_m : T(B) \to T(A), \quad n \mapsto m \cdot_B n.$$

Moreover if $A = B = C$, $\cdot_A$ endows $T(A,A)$ with a ring structure - we call $T(A,A)$ the **trivial source ring of** A.

Definition (Dual Homomorphism)

Given an (A,B)-bimodule M, taking its dual M^* induces a group homomorphism

$$(-)^* : T(A,B) \to T(B,A).$$

Definition (p-Permutation Equivalences)

For A, B direct summands of group algebras, denote $T^{\Delta}(A,B) \leq T(A,B)$ the subgroup consisting of p-permutation bimodules with (twisted) diagonal vertices.

Say $\gamma \in T^{\Delta}(A,B)$ is a p-**permutation equivalence** for A and B if γ^* satisfies

$$\gamma \cdot_B \gamma^* = [A], \quad \gamma^* \cdot_A \gamma = [B].$$

The set of p-permutation equivalences for A and B is denoted $T_o^{\Delta}(A,B)$.

A p-permutation equivalence induces an isomorphism $T(A) \cong T(B)$, and further, by a result of Boltje and Perepelitsky, p-permutation equivalences induce isotypies.

Remark

Not every $\gamma \in T^{\Delta}(A,B)$ for which there is a $\gamma' \in T^{\Delta}(B,A)$ such that $\gamma \cdot_B \gamma' = [A]$, $\gamma' \cdot_A \gamma = [B]$ satisfies $\gamma' = \gamma^*$. In general, the set of p-permutation equivalences is finite.

Theorem (Boltje, Perepelitsky)

Let A and B be direct summands of block algebras. If Γ is a splendid Rickard equivalence for A and B, then

$$\gamma_\Gamma := \sum_{i\in\mathbb{Z}}(-1)^i[\Gamma_i] \in T_o^\Delta(A,B),$$

is a p-permutation equivalence for A and B. Moreover, the two equivalences induce the same isotypy.

My Questions

- Given a p-permutation equivalence, is it induced by a splendid Rickard complex?
- If so, how can we construct its lift?

To begin, we'd like to work with examples. Unfortunately, examples of these equivalences are scarce.

However, we have one fruitful source of p-permutation equivalences, the unit group of the Burnside ring, $B(G)^\times$. This produces p-permutation *self-equivalences*, equivalences between RG and RG. As an group, $B(G)$ is a Grothendieck group for G-sets with respect to disjoint union.

Definition (Burnside Ring)

The Burnside Group $B(G)$ of G is the quotient of the free abelian group $\mathbb{Z}[I]$, with I the set of isomorphism classes of finite G-sets, by the subgroup S which is $\mathbb{Z}$-linearly generated by elements of the form

$$[X] + [Y] - [X \sqcup Y].$$

$B(G)$ is a ring under the multiplication

$$[X] \cdot [Y] := [X \times Y],$$

hence, $B(G)$ is **the Burnside Ring of** G. The elements of $B(G)$ are called **virtual** G**-sets**.

The Ghost Ring

1. (Burnside) $[X] = [Y] \iff |X^H| = |Y^H|$ for all $H \leq G \iff X \cong Y$ as G-sets.
2. Let $\mathcal{S}(G)$ denote a set of representatives of conjugacy classes of subgroups of G. Then $B(G)$ as a $\mathbb{Z}$-module has canonical basis given by
$$\{[G/K] : K \in \mathcal{S}(G)\}.$$
3. For any $H \leq G$, the fixed point function $|(-)^H| : B(G) \to \mathbb{Z}$ is a ring homomorphism. It follows that any $X \in B(G)$ can be equivalently considered a function
$$\phi_X : \mathcal{S}(G) \to \mathbb{Z}, \quad K \mapsto |X^K|.$$
Under this assignment, $B(G)$ is injectively embedded in $\mathrm{Hom}(\mathcal{S}(G), \mathbb{Z})$, the **ghost ring**, and
$$\mathbb{Q} \otimes_{\mathbb{Z}} B(G) \cong \mathrm{Hom}(\mathcal{S}(G), \mathbb{Q}).$$
The values of ϕ_X are its **marks**.

Definition (Bisets)

A (H, G)-**biset** X is a left H-set and right G-set for which the actions commute. Equivalently, it is a left $(H \times G^{op})$-set (in the notation of Bouc).

Bisets are "composable" in a compatible sense.

Definition (Biset Composition)

Given a (K, H)-biset X and (H, G)-biset Y, the **composition** of X and Y is the set of H-orbits on the set $X \times Y$ with right H-action defined by

$$(x, y) \cdot h = (x \cdot h, h^{-1} \cdot y).$$

It is denoted by $X \times_H Y$, and the orbit of (x, y) is written $(x,_H y)$. $X \times_H Y$ is a (K, G)-biset with action defined by

$$k \cdot (x,_H y) \cdot g := (k \cdot x,_H y \cdot g).$$

It is a straightforward exercise to show that R-linearizing the biset $X \times_H Y$ gives an isomorphism of (RK, RH)-bimodules

$$R[X \times_H Y] \cong R[X] \otimes_{RG} R[Y].$$

Definition (Biset Burnside Ring)

As a group, the **biset Burnside ring** $B(G,G)$ is defined as $B(G \times G^{op})$. It has multiplication defined by

$$[X] \cdot [Y] := [X \times_G Y].$$

Definition (Opposite Biset)

Given a (H,G)-biset X, its **opposite biset** X^{op} is X as a set, with (G,H)-biset structure given by

$$g \cdot x \cdot h := h^{-1} \cdot x \cdot g^{-1}.$$

This defines an involution on $B(G,G)$, given by

$$[X] \mapsto [X^{op}].$$

Notice that $T(RG,RG)$ also has an involution given by taking the dual,

$$[M] \mapsto [M^*] = [\mathrm{Hom}_R(M,R)].$$

Two Ring Homomorphisms

1. There exists a ring homomorphism $\widetilde{(-)} : B(G) \to B(G,G)$, induced by the assignment
$$[G/K] \mapsto [\widetilde{G/K}] = [(G \times G)/\Delta K].$$
The image of of this assignment is self-dual in $B(G,G)$, i.e. $[\widetilde{X}^{op}] = [\widetilde{X}]$. Therefore, this map is compatible with the involutions of the rings, where $B(G)$ has trivial involution.
2. $R[-] : B(G,G) \to T^{\Delta}(RG,RG)$ induced by R-linearizing a (G,G)-biset,
$$X \mapsto R[X]$$
is a ring homomorphism. This map is compatible with the involutions defined on each ring, i.e. if X is a (G,G)-biset,
$$R[X^{op}] \cong R[X]^*$$

Restricting these ring homomorphisms to their unit groups provides group homomorphisms of mulitiplicatively invertible elements. For $T^{\Delta}(RG,RG)$, this includes all p-permutation equivalences.

$B(G)^\times$ Generates p-Permutation Equivalences

- Since $B(G)$ embeds into $\mathrm{Hom}(\mathcal{S}(G), \mathbb{Z})$, $B(G)^\times$ embeds into $\mathrm{Hom}(\mathcal{S}(G), \{\pm 1\})$.
- From this, it follows that $B(G)^\times$ is an elementary abelian 2-group. In particular, every element is self-inverse.
- The image of $B(G)^\times$ under $\widetilde{(-)}$ is self-dual, therefore its image in $T(RG, RG)$ is as well.
- Given some $u \in B(G)^\times$, $R[\tilde{u}] \in T^\Delta(RG, RG)$ is self-inverse and self-dual, thus orthogonal, therefore it is a p-permutation equivalence!

Given a $u \in B(G)^\times$, if a splendid Rickard complex Γ_u descends to $R[\tilde{u}]$, we call $\Gamma_u \leftrightarrow u$ **splendid correspondents.**

$$\begin{tikzcd}
 & & \mathrm{Spl}(G,G) \arrow[d, squiggly] \\
B(G)^\times \arrow[r] \arrow[urr, dotted] & O(B(G,G)) \arrow[r] & T_o^\Delta(RG,RG)
\end{tikzcd}$$

First direction of research: given a finite group G, consider $B(G)^\times$. Does every element have a splendid correspondent?

Caveat

In general, $B(G)^\times$ is difficult to describe.

Example: If G has odd order, $|B(G)^\times| = 1 \iff G$ is solvable. In other words, classifying $B(G)^\times$ for odd order groups is equivalent to proving the Feit-Thompson theorem.

However, there are some easier cases to work with.

Theorem (Matsuda)

If G is abelian, $B(G)^\times$ is generated by the following set:

$$\{-[G/G]\} \cup \{[G/G] - [G/H] : [G:H] = 2\}.$$

Proposition (Linkelmann/M.)

If $u \leftrightarrow \Gamma_u$ *and* $v \leftrightarrow \Gamma_v$ *are splendid correspondents, then* $uv \leftrightarrow \Gamma_u \otimes_{RG} \Gamma_v$ *are splendid correspondents. Additionally* $-u \leftrightarrow \Gamma[1]$*.*

In particular, it suffices to only provide lifts for the generators of $B(G)^\times$.

Theorem (M.)

For any finite group G*,*

$$\Gamma = \cdots \to 0 \to RG \otimes_{RH} RG \xrightarrow{d} RG \to 0 \to \cdots$$

is a splendid correspondent to $u = [G/G] - [G/H] \in B(G)^\times$*, where* d *is the map induced by multiplication.*

As an immediate corollary by the result of Matsuda, in the case of G abelian, every unit has a splendid correspondent.

Let's sketch the proof.

- First, we wish to understand Γ^*. Here, there is a nice identification:

$$\begin{tikzcd}
\Gamma' = & 0 \arrow[r] & RG \arrow[r, "d'"] \arrow[d, "\cong"] & RG \otimes_{RH} RG \arrow[r] \arrow[d, "\cong"] & 0 \\
\Gamma^* = & 0 \arrow[r] & RG^* \arrow[r, "d^*"] & (RG \otimes_{RH} RG)^* \arrow[r] & 0
\end{tikzcd}$$

$$d' : g \mapsto \sum_{h \in [G/H]} gh \otimes h^{-1}.$$

  Hence the two chain complexes are isomorphic, and we may identify Γ^* with Γ'.

- Since for any $K \leq G$,

$$RG \otimes_{RK} RG \cong \mathrm{Ind}_{\Delta K}^{G \times G}(R),$$

  it follows that all terms in Γ are p-permutation modules with diagonal vertices.

- It remains to show $\Gamma \otimes_{RG} \Gamma' \simeq RG$ and $\Gamma' \otimes_{RG} \Gamma \simeq RG$. We show only the first, the second follows similarly. (A future goal is to prove that in this case, $\Gamma \otimes_{RG} \Gamma^* \simeq RG$ if and only if $\Gamma^* \otimes_{RG} \Gamma \simeq RG$. Rickard proves this fact in the case of block algebras only.)
- $\Gamma \otimes_{RG} \Gamma' \simeq RG$ if and only if $\Gamma \otimes_{RG} \Gamma'$ is a split chain complex and has homology concentrated in degree zero isomorphic to RG. The following lemmas use the fact that $(\Gamma \otimes_{RG} \Gamma^*)^* \cong \Gamma \otimes_{RG} \Gamma^*$ naturally.

Proposition (M.)

Let A and B be k-algebras, and let Γ be a chain complex of (B, A)-bimodules. Set $M = \Gamma \otimes_A \Gamma^*$. Then $d_n : M_n \to M_{n-1}$ splits (resp. injectively, resp. surjectively) if and only if $d_{-(n-1)} : M_{-(n-1)} \to M_{-n}$ splits (resp. surjectively, resp. injectively).

Proposition (M.)

Let A, B be k-algebras and Γ a chain complex of finitely generated A, B-bimodules. $H_n(\Gamma \otimes_A \Gamma^*) = \{0\}$ if and only if $H_{-n}(\Gamma \otimes_A \Gamma^*) = \{0\}$.

Proposition (M.)

Suppose Γ is a finite split chain complex of p-permutation (kH, kG)-bimodules such that the map $\gamma_{(-)} : Spl_{G,G} \to O(T(H,G))$ sends Γ to a p-permutation equivalence. If the homology of $\Gamma \otimes_{kG} \Gamma^*$ is concentrated in degree 0, then $H_0(\Gamma \otimes_{kG} \Gamma^*) = kH$.

- We have that $\Gamma \otimes_{RG} \Gamma'$, after the identification $RG \otimes_{RG} RG \cong RG$ is:

$$\begin{tikzcd}
kG \otimes_{kH} kG \arrow[d, "d\otimes\mathrm{id}"] \arrow[rr, "-\mathrm{id}\otimes d^*"] & & \\
kG \arrow[rr, "\mathrm{id}\otimes d^*"] & & kG \otimes_{kH} kG \otimes_{kH} kG \arrow[d, "d\otimes\mathrm{id}"] \\
& & kG \otimes_{kH} kG
\end{tikzcd}$$

By the previous propositions, it suffices to show that one of the two transition maps splits, and that the homology at either of the outer nonzero terms is zero. There is one more technical lemma that helps with this.

Theorem (M.)

Let G be a finite group, and let $L, H \leq G$, and $H' < H$. Suppose there exists a set S of double coset representatives of $L\backslash G/H$ and a set S' of double coset representatives of $L\backslash G/H'$ such that $S \subseteq S'$ and for all $g \in S$, $L \cap {}^gH = L \cap {}^gH'$. Then the map

$$\begin{aligned} d^o_{L,H,H'} : kG \otimes_{kL} kG \otimes_{kH} kG &\to kG \otimes_{kL} kG \otimes_{kH'} kG \\ a \otimes b \otimes c &\mapsto \sum_{g \in [H/H']} a \otimes bg \otimes g^{-1}c \end{aligned}$$

splits injectively, where $[H/H']$ denotes a set of coset representatives of H/H'.

$$\begin{tikzcd}
kG \otimes_{kH} kG \arrow[d, "d\otimes\mathrm{id}"] \arrow[dr, "-\mathrm{id}\otimes d^*"] & \\
kG \arrow[dr, "\mathrm{id}\otimes d^*"] & kG \otimes_{kH} kG \otimes_{kH} kG \arrow[d, "d\otimes\mathrm{id}"] \\
 & kG \otimes_{kH} kG
\end{tikzcd}$$

The map $-\mathrm{id} \otimes d^*$ is of this form and satisfies the conditions of the theorem, thus it splits injectively. It follows that the transition map splits injectively, and the result follows.

The case of $B(S_3)^\times$

$B(S_3)^\times$ has $\mathbb{F}_2$-dimension 3, and is generated by

$$\{-[S_3/S_3], [S_3/S_3] - [S_3/A_3], [S_3/S_3] - 2[S_3/C_2] + [S_3/1]\}.$$

The first unit has a trivial correspondent, and the second unit was covered in the previous case.

Theorem (M.)

$$\Gamma_u = kS_3 \otimes_k kS_3 \xrightarrow{d_1} kS_3 \otimes_{k\langle(12)\rangle} kS_3 \oplus kS_3 \otimes_{k\langle(13)\rangle} kS_3 \xrightarrow{d_0} kS_3$$

$$d_1 : a \otimes b \mapsto (a \otimes b, a \otimes b), \quad d_0 : (a \otimes b, c \otimes d) \mapsto ab - cd.$$

is a splendid correspondent to $[S_3/S_3] - 2[S_3/C_2] + [S_3/1]$.

Its dual may be shown to be isomorphic to:

$$\Gamma_u^* : kS_3 \otimes_k kS_3 \xleftarrow{d_1^*} kS_3 \otimes_{k\langle(12)\rangle} kS_3 \oplus kS_3 \otimes_{k\langle(13)\rangle} kS_3 \xleftarrow{d_0^*} kS_3$$

$$d_0^* : a \mapsto \left(\sum_{g \in [S_3/\langle(12)\rangle]} ag \otimes g^{-1}, - \sum_{g \in [S_3/\langle(13)\rangle]} ag \otimes g^{-1} \right)$$

$$d_1^* : (a \otimes b, c \otimes d) \mapsto \sum_{g \in \langle(12)\rangle} ag \otimes g^{-1} b + \sum_{g \in \langle(13)\rangle} cg \otimes g^{-1} d.$$

Then $\Gamma_u \otimes_{RG} \Gamma_u^*$ can be depicted as follows:

$$\begin{tikzcd}
kS_3 \otimes_k kS_3 \arrow[d, "d_1 \otimes \mathrm{id}"] \arrow[dr, "-\mathrm{id} \otimes d_0^*"] & & \\
\bigoplus_{i=1}^2 kS_3 \otimes_{kC_2} kS_3 \arrow[d, "d_0 \otimes \mathrm{id}"] \arrow[dr, "\mathrm{id} \otimes d_0^*"] & \bigoplus_{i=1}^2 kS_3 \otimes_k kS_3 \otimes_{kC_2} kS_3 \arrow[d, "d_1 \otimes \mathrm{id}"] \arrow[dr, "-\mathrm{id} \otimes d_1^*"] & \\
kS_3 \arrow[dr, "-\mathrm{id} \otimes d_0^*"'] & \bigoplus_{i=1}^4 kS_3 \otimes_{kC_2} kS_3 \otimes_{kC_2} kS_3 \arrow[d, "d_0 \otimes \mathrm{id}"] \arrow[dr, "\mathrm{id} \otimes d_1^*"] & kS_3 \otimes_k kS_3 \otimes_k kS_3 \arrow[d, "d_1 \otimes \mathrm{id}"] \\
 & \bigoplus_{i=1}^2 kS_3 \otimes_{kC_2} kS_3 \arrow[dr, "-\mathrm{id} \otimes d_1^*"'] & \bigoplus_{i=1}^2 kS_3 \otimes_{kC_2} kS_3 \otimes_k kS_3 \arrow[d, "d_0 \otimes \mathrm{id}"] \\
 & & kS_3 \otimes_k kS_3
\end{tikzcd}$$

Verifying that homology is concentrated in degree 0 follows from computing the homology of Γ_u, but there are technical obstructions to verifying splitness of the inner maps. We will not discuss those here.

Further Directions of Study:

- Generalize the S_3 construction to arbitrary units of the form $[G/G] - [G/H_1] - [G/H_2] + [G/K]$ - these are important in the classification of $B(G)^\times$ if G is a 2-group.
- Inflation: Given $N \leq H_1, \ldots, H_k \leq G$ with $N \trianglelefteq G$,

  $$\sum_{i=1}^{k} (-1)^{a_i} [G/H_i] \in B(G)^\times \iff \sum_{i=1}^{k} (-1)^{a_i} [(G/N)/(H_i/N)] \in B(G/N)^\times.$$

  Can we find some corresponding “inflation” construction on complexes?
- Attempt constructions based on the image of $u \in B(G)^\times$ in $\mathrm{Hom}(\Phi(G), \{\pm 1\})$ - certain properties of $B(G)$ are easily communicated only in terms of the ghost ring.
- Search for p-permutation equivalences which do *not* lift. While it is unlikely every p-permutation lifts, there are no known examples of ones which do not lift as of yet.

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