A Splendid Lift of Equivalences

Lifting p-Permutation Equivalences to Splendid Rickard Complexes

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Outline

1. Intro to representation theory - goal: introduce **vertices, sources, defect groups,** and **blocks**.
2. Module categories and equivalences - goal: introduce **Splendid Rickard Complexes** and **Broue's Abelian Defect Group Conjecture**.
3. Grothendieck Groups of p-permutation modules - goal: introduce p-**permutation equivalences** and their connection to splendid Rickard complexes.
4. Burnside rings - goal: explain how units of $B(G)$ induce p-permutation equivalences.
5. Discuss my initial results.

Definition (Representation)

Let G be a group, k be a field, $n \in \mathbb{N}$, and V a n-dimensional vector space over k. A **representation of G of degree n** is a group homomorphism

$$\rho : G \to \mathsf{Aut}_k(V) \cong \mathrm{GL}_n(k).$$

Example

1. Let $\omega_n = 2\pi/n$. A representation of $C_n = \langle\sigma\rangle$ over $\mathbb{C}$ is given by

$$\sigma \mapsto \begin{pmatrix} \cos\omega_n & -\sin\omega_n \\ \sin\omega_n & \cos\omega_n \end{pmatrix}.$$

2. A representation of S_3 can be described by how the elements permute the ordered set $(1,2,3)$, for example,

$$(12) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (13) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

This is an example of a **permutation representation**.

Remark

When G is finite, we have a dual notion of a representation given by considering modules over the group algebra kG.

1. Given a representation $\rho : G \to \mathrm{Aut}(M)$ over a k-vector space M, we can give M a kG-module structure by
$$\left(\sum_{g\in G} a_g \cdot g\right) m := \sum_{g \in G} a_g \cdot \rho(g)(m).$$
2. Conversely, given a finitely generated kG-module M, it is naturally a k-vector space. For every element $g \in G$, the map $m \to g \cdot m$ lies in $\mathrm{Aut}_k(M)$. This defines a representation:
$$\begin{aligned} \rho : G &\to \mathrm{Aut}_k(M) \\ g &\mapsto (m \mapsto g \cdot m). \end{aligned}$$

These constructions are functorial and mutually inverse, and when G is finite, we have an equivalence of categories ${}_{kG}\mathbf{mod} \cong \mathbf{Rep}_k(G)$.

For the rest of this talk (unless specified), G will be finite, and a "representation" will be a finitely generated kG-module.

Example

1. kG is itself a representation, the **regular representation**.
2. If X is a G-set, then $k[X]$, the k-linearization of X, is a kG-module via the group action, i.e.

$$g \cdot (k_1 x_1 + \cdots + k_n x_n) := k_1(g \cdot x_1) + \cdots + k_n(g \cdot x_n).$$

   This is a **permutation representation**.

Definition

A representation M, i.e. a kG-module, is:

- **irreducible** or **simple** if its only submodules are $\{0\}$ and M.
- **indecomposable** if M cannot be written as $M = M_1 \oplus M_2$ for two nonzero representations M_1, M_2.
- **projective** if M is a direct summand of a free module, i.e. a direct summand of $kG \oplus \cdots \oplus kG$ for some finite number of kGs.

In **modular** representation theory, we work specifically over fields of prime characteristic dividing the order of G. What goes wrong?

Working over $\mathbb{C}$

- Simple $\iff$ indecomposable.
- Finitely many simple/indecomposable representations, or "irreps."
- All representations decompose into a direct sum of irreps.
- All representations are projective, i.e. $\mathbb{C}G$ is semisimple.

Working over prime characteristic dividing $|G|$

- Simple $\not\iff$ indecomposable.
- There are, in general, an infinite number of indecomposable representations.
- Representations do not in general decompose into irreps.
- Irreps are in general not projective - in fact, kG has infinite global projective dimension.

Definition (p-modular systems)

When working over nonzero characteristic, we don't just work over k, we have three rings, a p-**modular system** $(K, \mathcal{O}, k)$ **large enough for** G which satisfy the following:

1. $\mathcal{O}$ is a complete discrete valuation ring, i.e. a PID with a unique maximal ideal m, that has a $|G|$th root of unity.
2. K is the the field of fractions of $\mathcal{O}$, and has characteristic 0.
3. k is the residue field of $\mathcal{O}$, i.e. $\mathcal{O}/m$, and has characteristic p.

Every $\mathcal{O}G$-module descends to a kG-module, and every kG-module lifts (rarely uniquely) to an $\mathcal{O}G$-module. Working over KG is the same as working over $\mathbb{C}G$!

For the rest of this presentation, we will assume $(K, \mathcal{O}, k)$ is a p-modular system large enough for G, and let $R \in \{\mathcal{O}, k\}$.

Example

Let ζ be a $\exp G$-th root of unity. A large enough p-modular system is

- $K = \mathbb{Q}_p(\zeta)$.
- $\mathcal{O} = \mathbb{Z}_p[\zeta]$.
- $|k| = p^f$ for some $f \in \mathbb{N}^+$.

Let $H \leq G$ be groups. One question to ask is, "how can we go between studying representations of G and representations of H?"

Induction and Restriction

1. Given a RG-module M we can turn it into a RH-module by restricting scalars to RH. This is denoted $\operatorname{Res}^G_H M$.
2. Given a RH-module M, we can extend it to a RG-module by extending scalars to RG. Formally,

$$\operatorname{Ind}^G_H M := RG \otimes_{RH} M.$$

Definition (Relative projectivity)

Say a RG-module M is **relatively H-projective** if there exists a RH module N such that M is isomorphic to a direct summand of $\operatorname{Ind}^G_H N$ (writ. $M \mid \operatorname{Ind}^G_H N$).

Remark

1. By the Krull-Schmidt theorem, this is a well-defined notion.
2. M is 1-projective $\iff$ M is projective and projective as a R-module.

Definition (Vertices and Sources)

If M is an indecomposable RG-module, then there are minimal subgroups $P \leq G$ such that M is P-projective. In fact, all such P are G-conjugate p-subgroups.

1. Any such P is a **vertex** of M.
2. In this case, one may find an indecomposable RP-module S such that $M \mid \operatorname{Ind}_P^G S$. Any such S is a P-**source** of M.

Any pair (P, S) is a **vertex-source pair** of M.

Trivial Source Modules

Indecomposable RG-modules with trivial source (a **trivial source module**) are of special interest.

1. M has trivial source if and only if it is a direct summand of a permutation module.
2. Let N be a direct sum of trivial source modules. Then $\operatorname{Res}_P^G N$ is a permutation module for some Sylow p-subgroup $P \leq G$.

We call a direct sum of trivial source modules a p-**permutation module.**
Fact: p-permutation kG-modules and p-permutation $\mathcal{O}G$-modules correspond *bijectively!*

Definition (Blocks)

The set of primitive idempotents $\{e_1, \ldots, e_k\}$ of $Z(\mathcal{O}G)$ gives a decomposition

$$\mathcal{O}G = \mathcal{O}Ge_1 \oplus \cdots \oplus \mathcal{O}Ge_k$$

of $(\mathcal{O}G, \mathcal{O}G)$-bimodules. This is the *unique* decomposition of $\mathcal{O}G$ into a direct sum of two-sided ideals of $\mathcal{O}G$. We call the $\mathcal{O}Ge_i$s the **blocks** of $\mathcal{O}G$, and the e_is are the **block idempotents** of $\mathcal{O}G$. The blocks are again $\mathcal{O}$-algebras.

The block idempotents are also the primitive idempotents of $Z(KG)$, giving a decomposition

$$KG = KGe_1 \oplus \cdots \oplus KGe_k,$$

and the reduction $\overline{e_i} \in Z(kG)$ gives the primitive idempotents of $Z(kG)$ and a decomposition

$$kG = kG\overline{e_1} \oplus \cdots \oplus kG\overline{e_k}.$$

The corresponding direct summands and idempotents are the blocks and block idempotents of KG and kG respectively.

Fact

Let RG have block idempotents $\{e_1, \cdots, e_k\}$. Then any RG-module M has direct sum decomposition

$$M = Me_1 \oplus \cdots \oplus Me_k.$$

If $e_i M = M$ for some e_i, then we say M_i "belongs to the block e_i."
In particular, every indecomposable RG-module belongs to a unique block.

Definition (Principal block)

The block that the **trivial representation** $M = R$ belongs to is called the **principal block**.

Definition (Defect group of a block)

Each block RGe can also be regarded as a $R(G \times G)$-module, and one may show it has vertex $\Delta P \leq G \times G$ for some p-subgroup $P \leq G$.
P is the **defect group** of the block e.

Let A be a R-algebra (for example, RG). We have the category ${}_A$**mod** of finitely generated left A-modules. One question to ask is when two categories are equivalent, and what such an equivalence implies.

Definition (${}_A$mod)

${}_A$**mod**, the category of finitely generated left A-modules, is the following data:

- The objects are finitely generated left A-modules.
- The morphisms are module homomorphisms.

If $A \cong B$, then there is an equivalence of categories ${}_A$**mod** $\cong$ ${}_B$**mod** as abelian categories. Is the converse true?

Example

No! For example ${}_A$**mod** $\cong$ ${}_{M_n(A)}$**mod** for any positive integer n.

An equivalence of module categories of this form is a **Morita equivalence**.

Theorem (Morita)

*${}_A$**mod** $\cong {}_B$**mod** if and only if there exist an (A, B)-bimodule M and a (B, A)-bimodule N such that*

- *$M \otimes_B N \cong A$ as (A, A)-bimodule,*
- *$N \otimes_B M \cong B$ as (B, B)-bimodules,*

and with M and N finitely generated projective as left and right modules.

In this case, the functors $M \otimes_B -$ and $N \otimes_A -$ induce equivalences of categories.

So if a Morita equivalence exists, it is induced by tensoring by suitable bimodules.

In the case of representations

In the case when A and B are symmetric algebras, such as group algebras or block algebras, M and N can be chosen such that $N = M^* = \mathrm{Hom}_R(M, R)$!

Let's see how we can generalize Morita theory to more complicated categorical constructions.

Definition (Category of Chain Complexes)

Given ${}_A$**mod**, we can its **category of chain complexes**, $Ch({}_A\textbf{mod})$.

1. Objects are chain complexes of A-modules, i.e. chains of the form

$$M = \cdots \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} M_{n-1} \xrightarrow{d_{n-1}} \cdots$$

   satisfying $d_{i-1} \circ d_i = 0$ for all $i \in \mathbb{Z}$.

2. Morphisms are chain complex homomorphisms, i.e. collections of maps $\{f_i\}$ that make the diagram commute:

$$\begin{tikzcd}
\cdots \arrow[r] & M_n \arrow[r, "d_n"] \arrow[d, "f_n"] & M_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & N_n \arrow[r, "d'_n"] & N_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

You may recall chain complexes arising from computation of singular homology of topological spaces - a topological space gives rise to a chain complex and a continuous function between spaces induces a map of chain complexes.

Definition (Homotopy)

There is a notion (exact definition omitted) of two morphisms of chain complexes being **homotopic**. Write $f \sim g$ if f and g are homotopic.

Fact: in the case of chain complexes arising from singular homology, two continuous functions of topological spaces are homotopic if and only if the induced morphisms of chain complexes are homotopic!

Definition (Homotopy Category)

Define the **homotopy category** of A-modules,

$$K({}_A\textbf{mod}) := Ch({}_A\textbf{mod})/\sim,$$

i.e. identify all homotopic maps together.

Example (Newly isomorphic objects)

If two chain complexes M, N have maps $f : M \to N$ and $g : N \to M$ such that $f \circ g \sim \mathrm{id}_N$ and $g \circ f = \mathrm{id}_M$, then $M \cong N$ in $K({}_A\textbf{mod})$, but not necessarily in $Ch({}_A\textbf{mod})$. We write $M \simeq N$, and say M and N are **homotopy equivalent**.

Now let's formulate a Morita equivalence for the homotopy category.

Morita: the homotopy version

If there exists a chain complex of finitely generated (A, B)-bimodules M and a chain complex of finitely generated (B, A)-bimodules N such that:

- $M \otimes_B N \simeq A$,
- $N \otimes_A M \simeq B$,

then $M \otimes_B -$ and $N \otimes_A -$ induce an equivalence of categories $K({}_A\mathbf{mod}) \cong K({}_B\mathbf{mod})$.

In the case of block algebras, like in the module scenario, we can take $N = M^*$, the dual complex.

Definition (Dual functor)

Given an (A, B)-bimodule M, its dual $M^* := \mathrm{Hom}_R(M, R)$ is a (B, A)-bimodule under the actions

$$(b \cdot f \cdot a)(m) = f(a \cdot m \cdot b).$$

This induces a functor

$$(-)^* : {}_A\mathbf{mod}_B \to {}_B\mathbf{mod}_A.$$

Jeremy Rickard posited it is reasonable to impose additional conditions:

Definition (Splendid Rickard Complex)

Let A and B be direct summands of RG and RH respectively. A bounded complex X of finitely generated (A, B)-bimodules is a **Splendid Rickard Complex** if:

1. All the terms of X are p-permutation modules with "twisted diagonal" vertices, i.e. of the form ΔP for a common subgroup $P \leq G, H$.
2. $X \otimes_B X^* \simeq A$.
3. $X^* \otimes_A X \simeq B$.

A **Splendid Rickard Equivalence between** A **and** B is the equivalence $K({}_A\textbf{mod}) \cong K({}_B\textbf{mod})$ induced by tensoring by X.

Why p-permutation modules?

1. The Brauer construction in this case induces splendid Rickard equivalences between block algebras of centralizers of subgroups of G and H.
2. In the case where G is a finite reductive group, the equivalence should be related to the l-adic cohomology of Deligne-Lusztig varieties.

Splendid stands for: "**SPL**it-**END**omorphism two-sided tilting complex of summands of permutaton modules **I**nduced from **D**iagonal subgroups."

What does a homotopy equivalence imply for the block algebras?

- Existence of collections of "perfect" isometries between character rings, called *isotypies* (ask John McHugh for more details).
- Isomorphic centers.
- Isomorphic Hochschild cohomology rings.
- Isomorphic Cartan-Brauer triangles.

The motivation of studying these equivalences comes from conjectures of Michel Broué.

Broué's Abelian Defect Group Conjecture

Let G be a finite group and $(K, \mathcal{O}, k)$ a p-modular system large enough for G. Let A be a block algebra of $\mathcal{O}G$, D the defect group of A, and set $H := N_G(D)$. Then by Brauer's first main theorem, there is a corresponding block algebra B of $\mathcal{O}H$ with the same defect group.

Conjecture: If D is abelian, then there exists a splendid Rickard equivalence between A and B.

A weaker formulation is as follows:

Alternative Formulation

Suppose G has an abelian Sylow p-subgroup. Set $H := N_G(P)$. Then there is a splendid Rickard equivalence between the principal blocks of $\mathcal{O}G$ and $\mathcal{O}H$.

These conjectures were originally stated for derived equivalences, but Rickard showed that under certain conditions, they can be strengthened to homotopy equivalences.

If X is a splendid Rickard equivalence for two block algebras $\mathcal{O}A, \mathcal{O}B$, it is straightforward to show that $k \otimes_{\mathcal{O}} -$ induces a splendid Rickard equivalence $k \otimes_{\mathcal{O}} X$ for the block algebras kA and kB.

However, lifts to $\mathcal{O}$ are generally not unique on the module level, and it is not clear that a splendid Rickard equivalence between kA, kB descended from $\mathcal{O}A, \mathcal{O}B$.

Theorem (Rickard)

Given a splendid Rickard equivalence Y for kA, kB over k, there is a unique splendid Rickard equivalence X for $\mathcal{O}A, \mathcal{O}B$ such that $Y = k \otimes_{\mathcal{O}} X$.

Motivated by this construction, Robert Boltje and Philipp Perepletsky introduced the notion of a p-permutation equivalence.

Definition (Grothendieck Group of p-permutation modules)

The **Grothendieck Group** $T(RG)$ **of** G (with respect to direct sums) is the quotient of the free abelian group $\mathbb{Z}[I]$, with I the collection of isomorphism classes of p-permutation RG-modules, by the subgroup S generated by elements of the form

$$[M] + [N] - [M \oplus N].$$

Remarks

1. The standard basis of $T(RG)$ is given by all indecomposable trivial source RG-modules.
2. If e is a block idempotent of RG, we can analogously define $T(RGe)$ and view it as a subgroup of $T(RG)$, via the projection $\omega \mapsto \omega e$. It has standard basis given by all indecomposable trivial source RG-modules belonging to e.
3. Further, we can see $T(RGe_1 \oplus \cdots \oplus RGe_k) = T(RGe_1) \oplus \cdots \oplus T(RGe_k)$. If $A = RGe_1 \oplus \cdots \oplus RGe_k$ is a direct summand of RG, we write $T(A)$ as shorthand for above.

Remark

If G and H are groups, we have an equivalence of categories

$$_{RG}\textbf{mod}_{RH} \cong {}_{R[G\times H]}\textbf{mod} \cong {}_{RG\otimes_R RH}\textbf{mod},$$

induced by the formula

$$g\cdot m\cdot h^{-1} \leftrightarrow (g,h)\cdot m \leftrightarrow (g\otimes h)\cdot m.$$

In this way, we can consider a bimodule $M \in {}_{RG}\textbf{mod}_{RH}$ a p-permutation bimodule if it is a p-permutation module in ${}_{R[G\times H]}\textbf{mod}$.

Definition (Grothendieck Group of p-permutation bimodules)

We define

$$T(RG,RH) := T(R[G\times H]).$$

If e and f are block idempotents of RG and RH respectively, set

$$T(RGe,RHf) := T(R[G\times H](e\otimes \overline{f})),$$

where $\overline{(-)}$ is the involution homomorphism induced by $g\mapsto g^{-1}$. If A is a direct summand of RG and B a direct summand of RH, we define $T(A,B)$ analogously.

Remark

Let A, B, C be direct summands of group algebras. Tensor products of trivial source modules are again trivial source modules, so the tensor product induces a bilinear map

$$T(A,B) \times T(B,C) \to T(A,C), \quad (m,n) \mapsto m \cdot_B n,$$

where $\cdot_B$ is the map induced by $\otimes_B$. Letting $C = R$ shows that any $m \in T(A,B)$ induces a group homomorphism

$$I_m : T(B) \to T(A), \quad n \mapsto m \cdot_B n.$$

Moreover if $A = B = C$, $\cdot_A$ endows $T(A,A)$ with a ring structure - a **trivial source ring**.

Definition (Dual homomorphism)

Given an (A,B)-bimodule M, taking its dual M^* induces a group homomorphism

$$(-)^* : T(A,B) \to T(B,A).$$

Definition (p-permutation equivalence)

For A, B direct summands of group algebras, denote $T^{\Delta}(A,B) \leq T(A,B)$ the subgroup consisting of p-permutation bimodules with (twisted) diagonal vertices.

Say $\gamma \in T^{\Delta}(A,B)$ is a p-**permutation equivalence** for A and B if γ^* satisfies

$$\gamma \cdot_B \gamma^* = [A], \quad \gamma^* \cdot_A \gamma = [B].$$

The set of p-permutation equivalences for A and B is denoted $T_o^{\Delta}(A,B)$.

A p-permutation equivalence induces an isomorphism $T(A) \cong T(B)$, and further, it was shown that like splendid Rickard equivalences, p-permutation equivalences induce isotypies!

Remark

Note that not every $\gamma \in T^{\Delta}(A,B)$ for which there is a $\gamma' \in T^{\Delta}(B,A)$ such that $\gamma \cdot_B \gamma' = [A]$, $\gamma' \cdot_A \gamma = [B]$ satisfies $\gamma' = \gamma^*$. In general, the set of p-permutation equivalences is finite, but the set of "invertible" elements may be infinite.

So how are these two levels of equivalences connected?

Theorem (Boltje, Perepletsky)

Let A and B be direct summands of block algebras. If Γ is a splendid Rickard equivalence for A and B, then

$$\sum_{i\in\mathbb{Z}}(-1)^i[\Gamma_i]\in T_o^{\Delta}(A,B),$$

is a p-permutation equivalence for A and B. Moreover, the two equivalences induce the same isotypy.

So splendid Rickard complexes induce p-permutation equivalences!

My Questions

- Given a p-permutation equivalence, is it induced by a splendid Rickard complex? i.e. does it have a *lift*?
- If so, what is a construction of its lift?

To begin, we'd like to work with examples. However, nice examples of these equivalences are hard to find.

One source of p-permutation equivalences is the unit group of the Burnside ring, which produces p-permutation self-equivalences, equivalences between RG and RG. This is a Grothendieck group for G-sets with respect to disjoint union.

Definition (Burnside Ring)

The Burnside Group $B(G)$ of G is the quotient of the free abelian group $\mathbb{Z}[I]$, with I the set of isomorphism classes of finite G-sets, by the subgroup S generated by elements of the form

$$[X] + [Y] - [X \sqcup Y].$$

$B(G)$ is a ring under the multiplication

$$[X] \cdot [Y] := [X \times Y],$$

hence, $B(G)$ is **the Burnside Ring of G**. The elements of $B(G)$ are called **virtual G-sets**.

Facts about $B(G)$

1. Every $U \in B(G)$ can be written uniquely as $U = [X] - [Y]$ for some G-sets X, Y.
2. (Burnside) $[X] = [Y] \iff |X^H| = |Y^H|$ for all $H \leq G \iff X \cong Y$ as G-sets.
3. Let $\mathcal{S}(G)$ denote a set of representatives of conjugacy classes of subgroups of G. Then $B(G)$ as a $\mathbb{Z}$-module has canonical basis given by
$$\{[G/K] : K \in \mathcal{S}(G)\}.$$
4. For any $H \leq G$, the fixed point function $|(-)^H| : B(G) \to \mathbb{Z}$ is a ring homomorphism. It follows that any $X \in B(G)$ can be equivalently considered a function
$$\phi_X : \mathcal{S}(G) \to \mathbb{Z}, \quad K \mapsto |X^K|.$$
Under this assignment, $B(G)$ is injectively embedded in Hom$(\mathcal{S}(G), \mathbb{Z})$, the **ghost ring**, and
$$\mathbb{Q} \otimes_{\mathbb{Z}} B(G) \cong \text{Hom}(\mathcal{S}(G), \mathbb{Q}).$$
The values of ϕ_X are its **marks**.

Definition (Bisets)

A (H, G)-**biset** X is a left H-set and right G-set for which the actions commute. Equivalently, it is a $(H \times G^{op})$-set.

Bisets are "composable" in a compatible sense.

Definition (Biset Composition)

Given a (K, H)-biset X and (H, G)-biset Y, the **composition** of X and Y is the set of H-orbits on the set $X \times Y$ with right H-action defined by

$$(x, y) \cdot h = (x \cdot h, h^{-1} \cdot y).$$

It is denoted by $X \times_H Y$, and the orbit of (x, y) is written $(x,_H y)$. $X \times_H Y$ is a (K, G)-biset with action defined by

$$k \cdot (x,_H y) \cdot g := (k \cdot x,_H y \cdot g).$$

In the case where K, H, G are the same group, this provides a sort of multiplication.

Definition (Biset Burnside Ring)

As a group, the **biset Burnside ring** $B(G, G)$ is defined as $B(G \times G^{op})$. It has multiplication defined by

$$[X] \cdot [Y] := [X \times_G Y].$$

Definition (Opposite Biset)

Given a (H, G)-biset X, its **opposite biset** X^{op} is X as a set, with (G, H)-biset structure given by

$$g \cdot x \cdot h := h^{-1} \cdot x \cdot g^{-1}.$$

This defines an involution on $B(G, G)$, given by

$$[X] \mapsto [X^{op}].$$

Remark

Notice that $T(RG, RG)$ also has an involution given by taking the dual, i.e.

$$[M] \mapsto [M^*] = [\mathrm{Hom}_R(M, R)].$$

Two Ring Homomorphisms

1. There exists a ring homomorphism $\widetilde{(-)} : B(G) \to B(G,G)$, induced by the assignment
$$[\widetilde{G/K}] = [(G \times G)/\Delta K].$$
The image of of this assignment is self-dual in $B(G,G)$, i.e. $[\widetilde{X}^{op}] = [\widetilde{X}]$.
2. $R[-] : B(G,G) \to T^\Delta(RG,RG)$ induced by R-linearizing a (G,G)-biset is a ring homomorphism. This map is compatible with the involutions defined on each ring, i.e. if X is a (G,G)-biset,
$$[R[X^{op}]] = [R[X]^*]$$

Restricting these ring homomorphisms to their unit groups provides group homomorphisms of mulitiplicatively invertible elements. For $T^\Delta(RG,RG)$, this includes all p-permutation equivalences.

Units generate p-permutation equivalences!

- Since $B(G)$ embeds into $\mathrm{Hom}(\mathcal{S}(G), \mathbb{Z})$, $B(G)^\times$ embeds into $\mathrm{Hom}(\mathcal{S}(G), \{\pm 1\})$.
- From this, it follows that $B(G)^\times$ is an elementary abelian 2-group! In particular, every element is self-inverse.
- Given some $u \in B(G)^\times$, $R[\tilde{u}] \in T^\Delta(RG, RG)$ is self-inverse and self-dual, therefore it is a p-permutation equivalence!

Given a $u \in B(G)^\times$, if a splendid Rickard complex Γ_u descends to $R[\tilde{u}]$, we call $\Gamma_u \leftrightarrow u$ **splendid correspondents.**

$$\begin{tikzcd}
 & & \mathrm{Spl}(G, G) \arrow[d, squiggly] \\
B(G)^\times \arrow[r] \arrow[urr, dotted] & O(B(G, G)) \arrow[r] & T_o^\Delta(RG, RG)
\end{tikzcd}$$

Remark

It is easy to compute $R[\tilde{u}]$. On transitive G-sets, $[G/H] \in B(G) \mapsto [RG \otimes_{RH} RG] \in T^\Delta(RG, RG)$.

First question: given some $B(G)^\times$, does every element have a splendid correspondent?

Caveat

In general, $B(G)^\times$ is difficult to describe.

Example: If G has odd order, $|B(G)^\times| = 1 \iff G$ is solvable. In other words, classifying $B(G)^\times$ for odd order groups is equivalent to proving the Feit-Thompson theorem.

However, there are some easier cases to work with.

Theorem (Matsuda)

If G is abelian, $B(G)^\times$ is generated by the following set:

$$\{-[G/G]\} \cup \{[G/G] - [G/H] : [G:H] = 2\}$$

Theorem (Miller)

If $u \leftrightarrow \Gamma_u$ and $v \leftrightarrow \Gamma_v$ are splendid correspondents, then $uv \leftrightarrow \Gamma_u \otimes_{RG} \Gamma_v$ are splendid correspondents. Additionally $-u \leftrightarrow \Gamma[1]$.

In particular, it suffices to only provide lifts for the generators of $B(G)^\times$.

Theorem (Miller)

For any finite group G,

$$\cdots \to 0 \to RG \otimes_{RH} RG \xrightarrow{d} RG \to 0 \to \cdots$$

is a splendid correspondent to $u = [G/G] - [G/H] \in B(G)^\times$, where d is the map induced by multiplication.

So in the case of G abelian, every unit has a splendid correspondent!

What are some tools we have to demonstrate a proposed complex Γ is indeed a splendid Rickard complex?

Lemmas (Miller):

$\Gamma \otimes_{RG} \Gamma^* \simeq RG$ if and only if $\Gamma \otimes_{RG} \Gamma^*$ is *split* and has homology concentrated in degree 0, isomorphic to RG.

1. Further, it suffices to show only half of the maps of $\Gamma \otimes_{RG} \Gamma^*$ are split.
2. If Γ descends to a p-permutation equivalence, one only has to verify exactness in the upper or lower part of $\Gamma \otimes_{RG} \Gamma^*$.

Moreover, there is a nice identification of Γ^* using the fact that every permutation module is self-dual.

The case of $B(S_3)^\times$

$B(S_3)^\times$ has $\mathbb{F}_2$-dimension 3, and is generated by

$$\{-[S_3/S_3], [S_3/S_3] - [S_3/A_3], [S_3/S_3] - 2[S_3/C_2] + [S_3/1]\}.$$

The first unit has a trivial correspondent, and the second unit was covered from before. The third unit u is tricky...

We propose the following complex:

$$\Gamma_u = kS_3 \otimes_k kS_3 \xrightarrow{d_1} kS_3 \otimes_{k\langle(12)\rangle} kS_3 \oplus kS_3 \otimes_{k\langle(13)\rangle} kS_3 \xrightarrow{d_0} kS_3$$

$$d_1 : a \otimes b \mapsto (a \otimes b, a \otimes b), \quad d_0 : (a \otimes b, c \otimes d) \mapsto ab - cd.$$

Its dual may be shown to be isomorphic to:

$$\Gamma_u^* : kS_3 \otimes_k kS_3 \xleftarrow{d_1^*} kS_3 \otimes_{k\langle(12)\rangle} kS_3 \oplus kS_3 \otimes_{k\langle(13)\rangle} kS_3 \xleftarrow{d_0^*} kS_3$$

$$d_0^* : a \mapsto \left(\sum_{g \in [S_3/\langle(12)\rangle]} ag \otimes g^{-1}, - \sum_{g \in [S_3/\langle(13)\rangle]} ag \otimes g^{-1} \right)$$

$$d_1^* : (a \otimes b, c \otimes d) \mapsto \sum_{g \in \langle(12)\rangle} ag \otimes g^{-1} b + \sum_{g \in \langle(13)\rangle} cg \otimes g^{-1} d.$$

Then $\Gamma_u \otimes_{RG} \Gamma_u^*$ can be depicted as follows:

$$\begin{tikzcd}
kS_3 \otimes_k kS_3 \arrow[d, "d_1 \otimes \mathrm{id}"] \arrow[dr, "-\mathrm{id} \otimes d_0^*"] & & \\
\bigoplus_{i=1}^2 kS_3 \otimes_{kC_2} kS_3 \arrow[d, "d_0 \otimes \mathrm{id}"] \arrow[dr, "\mathrm{id} \otimes d_0^*"] & \bigoplus_{i=1}^2 kS_3 \otimes_k kS_3 \otimes_{kC_2} kS_3 \arrow[d, "d_1 \otimes \mathrm{id}"] \arrow[dr, "-\mathrm{id} \otimes d_1^*"] & \\
kS_3 \arrow[dr, swap, "-\mathrm{id} \otimes d_0^*"] & \bigoplus_{i=1}^4 kS_3 \otimes_{kC_2} kS_3 \otimes_{kC_2} kS_3 \arrow[d, "d_0 \otimes \mathrm{id}"] \arrow[dr, "\mathrm{id} \otimes d_1^*"] & kS_3 \otimes_k kS_3 \otimes_k kS_3 \arrow[d, "d_1 \otimes \mathrm{id}"] \\
& \bigoplus_{i=1}^2 kS_3 \otimes_{kC_2} kS_3 \arrow[dr, swap, "-\mathrm{id} \otimes d_1^*"] & \bigoplus_{i=1}^2 kS_3 \otimes_{kC_2} kS_3 \otimes_k kS_3 \arrow[d, "d_0 \otimes \mathrm{id}"] \\
& & kS_3 \otimes_k kS_3
\end{tikzcd}$$

Verifying that homology is concentrated in degree 0 follows from computing the homology of Γ_u, but there are challenging obstructions to verifying splitness of the inner maps.

Further directions of study:

- Generalize the S_3 construction to arbitrary units of the form $[G/G] - [G/H_1] - [G/H_2] + [G/K]$ - these are important in the classification of $B(G)^\times$ if G is a 2-group.
- Inflation: Given $N \leq H_1, \ldots, H_k \leq G$ with $N \trianglelefteq G$,

  $$\sum_{i=1}^{k}(-1)^{a_i}[G/H_i] \in B(G)^\times \iff \sum_{i=1}^{k}(-1)^{a_i}[(G/N)/(H_i/N)] \in B(G/N)^\times.$$

  Can we find some corresponding "inflation" construction on complexes?
- Attempt constructions based on the image of $u \in B(G)^\times$ in Hom$(\Phi(G), \mathbb{Z})$ - certain properties of $B(G)$ are easily communicated only in terms of the ghost ring.
- Search for p-permutation equivalences which do *not* lift. While it is unlikely every p-permutation lifts, there are no known examples of ones which do not lift as of yet.