A Splendid Lift of Equivalences

Lifting p-Permutation Equivalences to Splendid Rickard Equivalences

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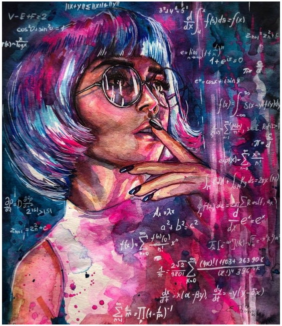
You're thinking: **"Can I have a translation of the title into human words?"**

Yes - but be patient! In these brief five minutes, my goals are to:

1. decipher the title, and
2. paint a picture of my research,



Right, E. (2020) The Beauty World of Mathematics

from scratch, with as few technical details as possible.

(unfortunately, there will still be some technical details, because math is hard)

In my field, **representation theory**, we study **representations** of **groups.**

We've got some definitions to make...

Groups

A **group** G is a set of things (elements), like numbers, that we can "add" together. (Plus a few additional rules)

Example

- The integers $(\ldots, -2, -1, 0, 1, 2\ldots)$ with addition.
- The set of ways you can twist a Rubik's cube.

Groups are one of the most fundamental objects in mathematics!

Representations

A **representation** is a way of "representing" groups with tables full of numbers that we can add and multiply, called **matrices.**

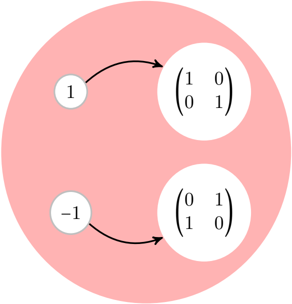
Example

Let's consider the group $G = \{1, -1\}$ (with multiplication). One representation of G is:

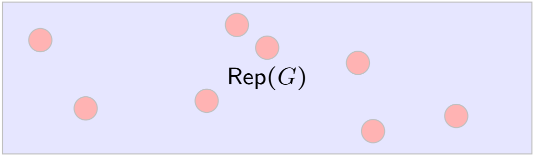
$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad -1 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Representations help make abstract groups more concrete - some groups can be hard to understand, but matrices are friendly!

Let's zoom out our perspective a bit. Here is our representation of G from before - think of it as a planet:



And here is a collection of a few representations of G - the solar system they live in:



Sometimes, we don't want to study just one representation, but rather, collections, or **spaces** (with some kind of underlying structure), **of representations.**

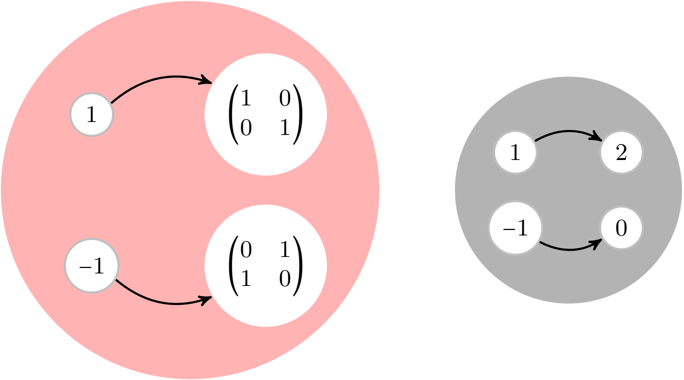
Grand spiral galaxy (NGC 1232).

But a collection of representations on its own can be very tough to understand. What else can we look at? There are many approaches...

Approach 1: Characters (looking at shadows!)

Each representation has a **character**, a function that pairs uniquely to the representation. You can think of it as a *shadow* of a representation!

Fact: Some collections of characters can be made into geometric spaces! These spaces are called **character rings**.

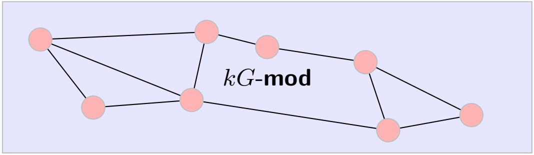


A representation and its character

Approach 2: Category Theory (abstracting everything!)

Category theory is a field of math that tries to generalize and unify mathematics with things called **categories**.

We can form a category out of the data in our set of representations (details omitted) - the **category of representations**.



A representation category

So now, we have tools to study **spaces of representations**:

1. Character rings
2. Representation categories

Important math question!

When do two things, or spaces of things (like collections of representations), share key properties?

To answer this, we look for **equivalences**, which tell us the things we're comparing are "essentially the same" in some way!

A Fruitful Idea: Equivalences of representation spaces can arise from character rings and categorical constructions!

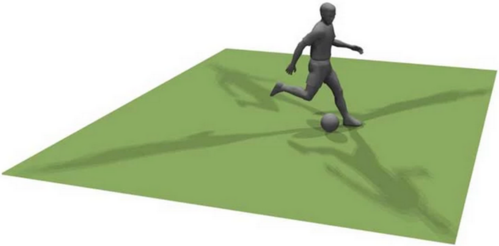
Now let's get back to that title!

"Lifting p-**Permutation Equivalences** to **Splendid Rickard Equivalences**"

Equivalence 1: p-Permutation Equivalences

p-**permutation equivalences** live on the **character** level. They are objects which form maps between collections of representation rings.

Think *"comparing shadows resulting from multiple light sources."*



Like how characters are the shadow of a representation, p-permutation equivalences may be the shadow of something greater...

Equivalence 2: Splendid Rickard Equivalences

A **splendid Rickard equivalence** looks like a chain of representations with a map (an arrow) between each link.

$$\Gamma = \cdots \to 0 \to kS_3 \otimes_k kS_3 \to kS_3 \otimes_{k\langle(12)\rangle} kS_3 \oplus kS_3 \otimes_{k\langle(13)\rangle} kS_3 \to kS_3 \to 0 \to \cdots$$

They form equivalences on categorical levels.

A Connection!

Fact: There is a construction turning any splendid Rickard equivalence into a p-permutation equivalence!

So, at least some p-permutation equivalences are **shadows** of splendid Rickard equivalences!

These two types of equivalences communicate very different properties. This connection unites two distant worlds... in one direction.

What about the other direction?

My Research

I have two main questions:

1. Are *all* p-permutation equivalences shadows of splendid Rickard equivalences?
2. How can we *lift* an arbitrary p-permutation equivalence to a splendid Rickard complex?

In short, I'm asking: **when are these two different types of equivalences actually equivalent?**

To summarize:

1. We care about **representations** of **groups**.
2. We don't just want to study representations, but also, **spaces of representations**.
3. Two tools for studying spaces of representations are **character rings** and **representation categories**.
4. In asking when spaces of representations share key properties, we look for **equivalences**.
5. A p-**permutation equivalence** is on the character ring level, and a **splendid Rickard equivalence** is on the categorical level.
6. Splendid Rickard equivalences **induce** p-permutation equivalences. My goal is to move in the other direction.

Thank you!!